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THE DIRICHLET AND THE WEIGHTED METRICS FOR THE SPACE OF KÄHLER METRICS

SIMONE CALAMAI AND KAI ZHENG

ABSTRACT. In this work we study the intrinsic geometry of the space of Kähler metrics under various Riemannian metrics. The first part is on the *Dirichlet metric*. We motivate its study, we compute its curvature, and we make links with the Calabi metric, the K-energy, the degenerate complex Hessian equation. The second part is on the weighted metrics, for which we investigate as well their geometric properties.

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1. INTRODUCTION

Let (M, ω) be a compact closed Kähler manifold of complex dimension n and let Ω be the Kähler class of ω ; namely, $\omega \in \Omega \in H_{dR}^2(M, \mathbb{R})$. The space of Kähler metrics \mathcal{H} corresponding to the Kähler class Ω is the subset of Ω containing all Kähler metrics. We will explain in next section that the space of Kähler metrics is

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in one to one correspondence both to the space of Kähler potentials (2.3) under the normalization condition (2.2), and to the space of volume conformal factors (2.1) and also to the space of Kähler one-forms (3.1).

In [6], Calabi proposed a variational problem on minimizing the Calabi energy in the space of Kähler metrics. Any minimizer is called an extremal metric. Calabi's extremal metrics generalize the constant scalar curvature Kähler metrics; as proved by Calabi in [4, page 99, Theorem 4]), whenever a space of Kähler metrics \mathcal{H} admits an extremal Kähler metric, then the vanishing of the Calabi-Futaki invariant is equivalent to the existence of a constant scalar curvature Kähler metric in \mathcal{H} . The underlying geometric structure of the moduli space \mathcal{H} has been extensively studied in recent years. In this paper, we study the intrinsic geometry of the space of Kähler metrics under various Riemannian metrics.

The most studied Riemannian metric in \mathcal{H} is the *Mabuchi metric* which is defined in Mabuchi [26], Donaldson [21] and Semmes [27] independently. Under this metric, \mathcal{H} becomes a non-positively curved infinite-dimensional symmetric space. Semmes [27] pointed out that the geodesic equation in \mathcal{H} is a homogeneous complex Monge-Ampère equation. Donaldson [21] conjectured that \mathcal{H} endowed with the Mabuchi metric is geodesically convex and is a metric space and pointed out the intensive relation between the geodesics of \mathcal{H} and the existence (through geodesic stability), uniqueness (through convexity along the geodesics) of the extremal metrics. We refer readers to our paper [9] for complete references. In this paper [9], we find geometric conditions on the Dirichlet boundary values with less regularity which assure the existence and uniqueness of $C^{1,1}$ geodesic segments.

A second Riemannian structure on the space of volume conformal factors (2.1) was hinted in [6] and recently developed in detail by the first author in [8]; it is called *the Calabi metric* (2.6). It enjoys nice properties, such as a positive constant sectional curvature (while the Mabuchi metric has non-constant non-positive sectional curvature), and it induces a metric space structure on \mathcal{H} (as well as the Mabuchi metric does). Moreover, the Calabi metric endows \mathcal{H} with a richer geometry than the Mabuchi one; both for the Cauchy and the Dirichlet problems, there are explicit, real analytic solutions. The work done in [8] immediately lead a good deal of advances on the Calabi metric. In [18], it is shown that the restriction of the Ebin metric [22], defined on the space \mathcal{M} of all Riemannian metrics, to the space \mathcal{H} is precisely the Calabi metric, and the metric completion of \mathcal{H} endowed with the Calabi metric is explicitly computed; in [19] Calabi metric was generalized in order to make the space \mathcal{H} as totally geodesic as possible in \mathcal{M} endowed with the Ebin metric.

In Section 3, we introduce a third way to describe the space of Kähler metrics, which is from the point of view of differential forms. In the space of Kähler one-forms (3.1), we define the Dirichlet metric as a canonical L^2 -inner product and see how it induces an almost complex structure naturally (cf. Remark 3.2).

A motivation for the study of this metric comes from [17]; there, Chen and the second author showed that the pseudo-Calabi flow is the gradient flow of the K -energy when \mathcal{H} is endowed precisely with the Dirichlet metric. In the same paper, they also developed a geometric method based on \mathcal{H} and proved the stability of the pseudo-Calabi flow near a cscK metric. In this paper, we prove its counterpart in the variational problem: the K -energy is convex at a point which corresponds to a constant scalar curvature Kähler metric (cf. Proposition 3.23). We also present

a formulation for the geodesic equation and we present the conjecture that the regularity of geodesics (if they exist) is greater than the $C^{1,1}$ regularity of Chen's geodesics and smaller than the C^ω regularity of Calabi's geodesics.

Now let us focus on the "shape" of the space \mathcal{H} . It is well known by the work of Mabuchi [26], that the Mabuchi metric has non-positive sectional curvature. On the other hand the Calabi metric, as shown in [8], has positive constant sectional curvature. When M is a Riemann surface, the space \mathcal{H} , endowed with the Dirichlet metric, becomes flat. For manifolds of higher dimension, it is not known whether \mathcal{H} endowed with the Dirichlet metric is flat. Calabi suggests that the sectional curvature of the Dirichlet metric should be between the sectional curvature of the Mabuchi metric and of the Calabi metric. In this paper, we first find out the explicit formula of the sectional curvature of the Dirichlet metric. Then we prove that for a two plane spanned by tangent vectors $\psi_1, \psi_2 \in T_\phi \mathcal{H}$, the sectional curvature of the Dirichlet metric is bounded above and below by a constant K which depends only on the point ϕ and on one generator ψ_1 (cf. Theorem 3.9).

We find out the sufficient conditions to determine the signature of the sectional curvature, they are the degenerate complex Hessian equations (see Remark 3.19). This leads us to conjecture that unlike the case when M is a Riemann surface, the sectional curvature is not flat.

In the second part of the paper, we introduce a family of Riemannian metrics on the space \mathcal{C} of volume conformal factors, and we call them weighted metric. In particular, the weighted metric whose weight function χ is $\chi(x) = x^{-1}$, is precisely the Calabi metric. For each of the weighted metrics, we compute its sectional curvature (Theorem 4.7) and its geodesic equation (Proposition 4.11). We call the metric corresponding to the weight function $\chi(x) = 1$ the constant weight metric. The constant weight metric enjoys several nice properties.

- It is a flat Riemannian metric (cf. Corollary 5.3);
- Its geodesics are smooth lines (cf. Theorem 5.5);
- The K -energy functional is convex along the geodesic in the first Chern class when $C_1 \leq 0$.

When $C_1 \leq 0$, these properties enable us to give an alternative proof of the uniqueness of the Kähler-Einstein metrics (cf. Theorem 5.9). We also find a new functional and we prove that its gradient flow also convergence to the Kähler-Einstein metrics (cf. Theorem 5.10).

2. NOTATIONS AND DEFINITIONS

With M, ω and Ω as in the Introduction, recall that the space of Kähler metrics is

$$(2.1) \quad \mathcal{H} = \{\omega | \omega \text{ is a Kähler metric in } \Omega\}.$$

In the given Kähler class Ω the total volume of M is a fixed constant that only depends on Kähler class, and we call it V in the remainder.

Choose a Kähler metric ω_1 in the Kähler class; according to the $\partial\bar{\partial}$ -lemma, for any smooth Kähler metric, there exists a unique (up to a constant) smooth function φ (called Kähler potential) such that $\omega_1 = \omega + i\partial\bar{\partial}\varphi$. So in order to assure the uniqueness of the Kähler potential, we require a normalization condition by means

of the following functional on \mathcal{H}

$$(2.2) \quad D_\omega(\varphi) = \sum_{i=0}^n \frac{n!}{(i+1)!(n-i)!} \int_M \varphi \omega^{n-i} \wedge (\partial\bar{\partial}\varphi)^i.$$

Since its first variation is $\int_M \dot{\varphi} \omega_\varphi^n$, we see that the Monge-Ampère operator is the Euler-Lagrange operator of D_ω .

Then the space of Kähler metrics is one to one correspondence to the space of Kähler potentials which is defined to be the set containing all the Kähler potentials with the following normalization condition, i.e.

$$(2.3) \quad \mathcal{H} = \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega + i\partial\bar{\partial}\varphi > 0, D_\omega(\varphi) = 0\}.$$

Whenever we have an element $\alpha \in C^1([0, 1] \times M, \mathbb{R})$ such that $\alpha(t, \cdot) \in \mathcal{H}$, for any $t \in [0, 1]$, then we call it a *differentiable curve*, and we write it simply as $\alpha : [0, 1] \rightarrow \mathcal{H}$ when no confusion arises. By means of the characterization (2.3) and of differentiable curves, the tangent space of \mathcal{H} at a point ϕ can be expressed as

$$T_\phi \mathcal{H} = \left\{ \psi \in C^\infty(M, \mathbb{R}) \mid \int_M \psi \frac{\omega_\phi^n}{n!} = 0 \right\}.$$

In the remainder of the paper, we will need to consider differentiable curves of tangent vectors, in the following sense. Given a differentiable curve α as before, we will consider a map $\beta \in C^1([0, 1] \times M, \mathbb{R})$ such that, for any $t \in [0, 1]$, then $\beta(t, \cdot) \in T_{\alpha(t, \cdot)} \mathcal{H}$.

The Mabuchi metric in \mathcal{H} is defined as follows; for any two tangent vectors ψ_1, ψ_2 in the tangent space of \mathcal{H} at ϕ , it is given by

$$(2.4) \quad Ma_\phi(\psi_1, \psi_2) = \int_M \psi_1 \psi_2 \frac{\omega_\phi^n}{n!}.$$

In the given Kähler class Ω , the corresponding volume forms form a conformal class. This is the second description of the space of Kähler metrics.

Definition 2.1. The *space of volume conformal factors* is given by the following space

$$\mathcal{C} := \left\{ F \in C^\infty(M, \mathbb{R}) \mid F > 0, \int_M F \omega^n = \int_M \omega^n \right\}.$$

The notions of differentiable curves, tangent vectors, curves of tangent vectors on \mathcal{C} are in exactly the same vein as it was for the space \mathcal{H} . For any fixed conformal factor $F \in \mathcal{C}$, we have that the tangent space of \mathcal{C} at F is

$$(2.5) \quad T_F \mathcal{C} = \left\{ G \in C^\infty(M, \mathbb{R}) \mid \int_M G \omega^n = 0 \right\}.$$

The Monge-Ampère map, defined by

$$MA : \mathcal{H} \longrightarrow \mathcal{C}$$

$$\phi \mapsto MA(\phi) := F(\phi) = \left(\frac{\omega_\phi^n}{\omega^n} \right),$$

is, by the positive answer to *the Calabi volume conjecture* [29], a diffeomorphism between \mathcal{H} and \mathcal{C} . Its differential at $\phi \in \mathcal{H}$ is

$$MA_*[\phi] : T_\phi \mathcal{H} \longrightarrow T_F \mathcal{C}$$

$$\psi \mapsto G(\psi) := \left(\frac{\omega_\phi^n}{\omega^n} \right) \Delta_\phi \psi ,$$

where the expression of Δ_ϕ in a local coordinate system (U, z^1, \dots, z^n) is $\Delta_\phi \psi = g_\phi^{j\bar{k}} \psi_{j\bar{k}}$, and g_ϕ is the Kähler metric corresponding to the Kähler form ω_ϕ .

The Calabi metric is defined for any $\psi_1, \psi_2 \in T_\phi \mathcal{H}$,

$$(2.6) \quad Ca_\phi(\psi_1, \psi_2) := \int_M \Delta_\phi \psi_1 \Delta_\phi \psi_2 \frac{\omega_\phi^n}{n!} .$$

To be precise, Mabuchi computed that, if $\phi = \phi(s, t)$ is a smooth two parameter family of curves in the space of Kähler metrics \mathcal{H} , and the corresponding two parameter families of curves of tangent vectors ϕ_t, ϕ_s along ϕ are \mathbb{R} -linearly independent, then the sectional curvature of the plane spanned by ϕ_s and ϕ_t is expressed in terms of their Poisson bracket as

$$\{\phi_s, \phi_t\}_\phi = \frac{\sqrt{-1}}{2} \left(g^{i\bar{j}} \frac{\partial \phi_s}{\partial z^i} \frac{\partial \phi_t}{\partial z^{\bar{j}}} - g^{i\bar{j}} \frac{\partial \phi_t}{\partial z^i} \frac{\partial \phi_s}{\partial z^{\bar{j}}} \right) = \text{Im}(\partial \phi_s, \bar{\partial} \phi_t)_\phi .$$

The expression of the sectional curvature K_M for the Mabuchi metric is therefore

$$K_M(\phi_s, \phi_t)_\phi = - \frac{\int_M \text{Im}(\partial \phi_s, \bar{\partial} \phi_t)_\phi^2 \frac{\omega_\phi^n}{n!}}{\sqrt{\int_M \phi_s^2 \frac{\omega_\phi^n}{n!}} \sqrt{\int_M \phi_t^2 \frac{\omega_\phi^n}{n!}} - \int_M \phi_s \phi_t \frac{\omega_\phi^n}{n!}} .$$

We read off the above equation that $K_M \leq 0$ for all the linearly independent sections ϕ_s, ϕ_t . On the other side, the first author proved that, for any linearly independent sections ϕ_s, ϕ_t the sectional curvature for the Calabi metric K_C is

$$K_C(\phi_s, \phi_t) = \frac{1}{4V} ;$$

here V is the volume of the manifold M . Eugenio Calabi conjectured that the sectional curvature for the Dirichlet metric is bounded above and below respectively by the Mabuchi and the Calabi sectional curvatures. In [8], the first author proved that if M has complex dimension one, then for any linearly independent sections ϕ_s, ϕ_t , the sectional curvature for the Dirichlet metric vanishes;

$$K_D(\phi_s, \phi_t) = 0 .$$

As a corollary, for closed Riemann surfaces we have that

$$K_{\text{Mabuchi}} \leq K_{\text{Dirichlet}} < K_{\text{Calabi}} .$$

One aim of the present paper is to take a step further towards the above bounds then the complex dimension of the manifold M is bigger than one. (cf. Subsection 3.2.)

3. THE DIRICHLET METRIC

We now want to introduce the Dirichlet metric. We shall use the point of view of differential forms. In the present section we are going to make repeatedly use of the Einstein notation.

Definition 3.1. We define the following space

$$(3.1) \quad \mathcal{A} := \{d\phi \mid \omega_\phi \text{ is a Kähler metric in } \mathcal{H}\}.$$

We note that the space \mathcal{A} is a \mathbb{R} -convex, open subset in the space of real 1-forms; indeed, this follows from the fact that the space \mathcal{H} is \mathbb{R} -convex, that is if ϕ_0, ϕ_1 are in \mathcal{H} , then for any $t \in [0, 1]$ also $(1-t)\phi_0 + t\phi_1$ is in \mathcal{H} .

Lemma 3.1. *The space \mathcal{A} is in one-to-one correspondence with the space of Kähler metrics \mathcal{H} .*

Proof. Consider the map $\Phi : \mathcal{H} \rightarrow \mathcal{A}$, which maps a Kähler metric ω_ϕ to the element $d\phi \in \mathcal{A}$. The map is well defined. Indeed, if two Kähler metrics ω_{ϕ_1} and ω_{ϕ_2} coincide, then they differ by an additive real constant, say $\phi_1 = \phi_2 + C$. Thereby, we have $d\phi_1 = d\phi_2$. The map is surjective by definition of the space \mathcal{A} . Concerning the injectivity, if we have $d\phi_1 = d\phi_2$, then we also have $\partial\bar{\partial}\phi_1 = \partial\bar{\partial}\phi_2$ and thereby the corresponding Kähler metrics ω_{ϕ_1} and ω_{ϕ_2} coincide. \square

As an application of Lemma 3.1, we can define differentiable curves on \mathcal{A} as the image under the map Φ of the already defined differentiable cruves on \mathcal{H} . And using differentiable curves, or just differentiating the map Φ , we conclude that the tangent space of \mathcal{A} at a point $d\phi$ is given by

$$T_{d\phi}\mathcal{A} = \{d\psi \mid \psi \in C^\infty(M, \mathbb{R})\},$$

that is, it is the space of exact real 1-forms on M .

Remark 3.2. *The infinite dimensional Riemannian manifold \mathcal{A} carries a natural almost complex structure J defined as*

$$J\partial\psi = i\partial\psi; J\bar{\partial}\psi = -i\bar{\partial}\psi.$$

We are ready to present a central notion of this paper.

Definition 3.2. Consider a point $d\phi \in \mathcal{A}$ and two tangent vectors $d\psi, d\chi \in T_{d\phi}\mathcal{A}$; let g_ϕ be the Kähler metrics corresponding to the element $d\phi \in \mathcal{A}$. Consider a local coordinate chart (U, z_1, \dots, z_n) on M . Then, let the pairing $(d\psi, d\chi)_{g_\phi}$ be defined, at a point $p \in U$, by

$$(d\psi, d\chi)_{g_\phi} = g_\phi^{i\bar{j}}(\psi_i\chi_{\bar{j}} + \psi_{\bar{j}}\chi_i),$$

where the Einstein notation is used. Now, the Dirichlet metric on $T_{d\phi}\mathcal{A}$ can be given as

$$Di_{d\phi} \langle d\psi, d\chi \rangle := \langle d\psi, d\chi \rangle_{g_\phi} = \int_M (d\psi, d\chi)_{g_\phi} \frac{\omega_\phi^n}{n!}.$$

As our goal is to prove the features of the geometry arising from the Dirichlet metric, it is natural to start with the study of the Levi-Civita covariant derivative of that metric.

3.1. Levi-Civita covariant derivative. The Levi-Civita covariant derivative on the space \mathcal{H} endowed with a Riemannian structure is, analogously to the case of standard Riemannian geometry, a way of differentiating tangent vector fields along curves on \mathcal{H} . For the precise definition of the Levi-Civita covariant derivative, we refer to [8, Definition 7]. As remarked in [21] its existence needs to be proved in our infinite dimensional environment, unlike the finite dimensional Riemannian geometry theory. The proof that the Levi-Civita covariant derivative for the Dirichlet metric exists can be found in [8]. Here, for the reader's convenience, we are going to represent those results with the slightly different approach of the previous section. To start off, we introduce a notation which will be very useful here and in the remainder of the paper.

Definition 3.3. Fix an element $d\phi \in \mathcal{A}$ and its corresponding Kähler form ω_ϕ . Let $f \in C^\infty(M, \mathbb{R})$ be a smooth real valued function. We define the $(1, 1)$ differential form $C[f]$ by $C[f] := (\Delta_\phi f)\omega_\phi - i\partial\bar{\partial}f$, where we recall that in a coordinate chart (U, z^1, \dots, z^n) , the expressions of the Kähler form ω_ϕ and of its corresponding Kähler metric g_ϕ are $g_\phi = g_{a\bar{b}}dz^a \otimes d\bar{z}^{\bar{b}}$ and $\omega_\phi = g_{a\bar{b}}dz^a \wedge d\bar{z}^{\bar{b}}$.

Lemma 3.3. Fix an $f \in C^\infty(M, \mathbb{R})$; after raising the indices of the differential form $C[f]$, and fixing any index i , we have, in a normal coordinate chart,

$$\sum_{j=1}^n (C[f]^{i\bar{j}})_{,\bar{j}} = 0.$$

Proof. Fix a point p in a normal coordinate chart (U, z_1, \dots, z_n) . For simplicity, we write g instead of g_ϕ . The operation of raising the indexes, when applied to $C[f]$ gives, for the component i, \bar{j} , the following output, (where for simplicity we write Δ instead of Δ_ϕ)

$$C[f]^{i\bar{j}} = \left[(\Delta f)g^{i\bar{b}}g_{a\bar{b}}g^{a\bar{j}} - g^{i\bar{b}} \frac{\partial^2 f}{\partial z^a \partial \bar{z}^{\bar{b}}} g^{a\bar{j}} \right].$$

Now, when we differentiate it with respect to $\frac{\partial}{\partial z^{\bar{j}}}$, we repeatedly use that, for any indexes a, b , then in normal coordinates there holds $\frac{\partial}{\partial z^{\bar{j}}} g^{a\bar{b}} = 0$; we get that

$$(3.2) \quad \frac{\partial}{\partial z^{\bar{j}}} \left[(\Delta f)g^{i\bar{b}}g_{a\bar{b}}g^{a\bar{j}} \right] = \left[\frac{\partial}{\partial z^{\bar{j}}} (\Delta f)g^{i\bar{b}} \right] g_{a\bar{b}}g^{a\bar{j}} = \left(\Delta \frac{\partial}{\partial z^{\bar{j}}} f \right) g^{i\bar{b}}g_{a\bar{b}}g^{a\bar{j}},$$

while

$$(3.3) \quad \frac{\partial}{\partial z^{\bar{j}}} \left[g^{i\bar{b}} \frac{\partial^2 f}{\partial z^a \partial \bar{z}^{\bar{b}}} g^{a\bar{j}} \right] = g^{i\bar{b}} \frac{\partial^3 f}{\partial z^a \partial \bar{z}^{\bar{b}} \partial z^{\bar{j}}} g^{a\bar{j}}.$$

Now, performing a summation with respect to the indexes a, b, j , we find that the two addenda (3.2) and (3.3) of $C[f]$ coincide. This completes the proof of the lemma. \square

Another formula which will be useful in the remainder is the derivative of the Laplacian operator.

Definition 3.4. Let us consider a pair of $(1, 1)$ -differential forms S, T and a one-covariant tensor α . Suppose that in a same coordinate chart (U, z^1, \dots, z^n) there

holds $T = T_{j\bar{k}} dz^j \wedge dz^{\bar{k}}$, $S = S_{j\bar{k}} dz^j \wedge dz^{\bar{k}}$ and $\alpha = \alpha_j dz^j + \alpha_{\bar{j}} dz^{\bar{j}}$. We define the function $S * T$ at a point $p \in U$ as

$$S * T := g_{\phi}^{j\bar{k}} g_{\phi}^{p\bar{q}} S_{j\bar{q}} T_{p\bar{k}}.$$

Notice that the operation $S * T$ is symmetric in S and T .

We also need to label as well by $*$ the following contraction of a twice-covariant tensor T and a one-covariant tensor α . The one-covariant tensor $T * \alpha$ is defined, at a point $p \in U$, as

$$(3.4) \quad T * \alpha = g_{\phi}^{a\bar{b}} T_{a\bar{k}} \alpha_{\bar{b}} dz^{\bar{k}} + g_{\phi}^{a\bar{b}} T_{\bar{k}b} \alpha_a dz^k.$$

Remark 3.4. Consider a two parameter family of curves on \mathcal{H} ; namely, $\phi = \phi(s, t) : [0, 1] \times [0, 1] \rightarrow \mathcal{H}$. Thus we have the formula

$$(3.5) \quad \frac{\partial}{\partial t} \left(\Delta_{\phi} \frac{\partial \phi}{\partial s} \right) = \Delta_{\phi} \frac{\partial^2 \phi}{\partial s \partial t} - \partial \bar{\partial} \frac{\partial \phi}{\partial t} * \partial \bar{\partial} \frac{\partial \phi}{\partial s}.$$

Definition 3.5. Fix a smooth curve $\phi = \phi(t) : [0, 1] \rightarrow \mathcal{H}$; also, let $\psi = \psi(t)$ be a smooth curve of tangent vectors along the curve ϕ , with $\psi(t) \in T_{\phi(t)} \mathcal{H}$ for any $t \in [0, 1]$. Write ϕ_t for $\frac{\partial \phi}{\partial t}$. Then, let $f[\phi_t, \psi] \in C^{\infty}(M, \mathbb{R})$ be such that $\Delta_{\phi} f[\phi_t, \psi] := \partial \bar{\partial} \psi * \partial \bar{\partial} \phi_t - \Delta_{\phi} \psi \cdot \Delta_{\phi} \phi_t$. Notice that, by the above remark, the function $f[\phi_t, \psi]$ is well defined.

As we are dealing with real differential forms, by means of the Hodge theory we decompose any real 1-form α as

$$\alpha = da + d^* \beta + h_{\alpha},$$

where $a \in C^{\infty}(M, \mathbb{R})$, β is a real 2-form and h_{α} is the harmonic part of α ; that is, we have $dh_{\alpha} = 0$ and $d^* h_{\alpha} = 0$. We also shall use the following notation for the natural projections;

$$(3.6) \quad \pi_d(\alpha) = da, \quad \pi_{d^*}(\alpha) = d^* \beta, \quad \pi_{\text{harm}}(\alpha) = h_{\alpha}.$$

The next result is preliminary for the main one of this section.

Lemma 3.5. Let $d\phi(t)$ be a smooth curve in the space \mathcal{A} and let $d\psi = d\psi(t)$ be a smooth curve of tangent vectors along the curve $d\phi$, with $d\psi(t) \in T_{d\phi(t)} \mathcal{A}$ for any $t \in [0, 1]$. Then we have

$$(3.7) \quad \int_M (\partial \bar{\partial} \psi * \partial \bar{\partial} \phi_t - \Delta_{\phi} \psi \cdot \Delta_{\phi} \phi_t) \frac{\omega_{\phi}^n}{n!} = 0;$$

also we have, in the notation of Definition (3.4),

$$(3.8) \quad df[\phi_t, \psi] = \pi_d \left(\frac{1}{2} C[\phi_t] * d\psi \right).$$

Proof. Concerning (3.7), it is readily computed as

$$0 = -\frac{\partial}{\partial t} \int_M \Delta_{\phi} \psi \frac{\omega_{\phi}^n}{n!} = \int_M (\partial \bar{\partial} \psi * \partial \bar{\partial} \phi_t - \Delta_{\phi} \psi \cdot \Delta_{\phi} \phi_t) \frac{\omega_{\phi}^n}{n!}.$$

About (3.8) let us call for convenience

$$\alpha = \frac{1}{2} C[\phi_t] * d\psi;$$

then, let \tilde{f} be any real valued smooth function on M and let us consider the integral

$$\int_M \tilde{f} \cdot d^* \alpha \frac{\omega_\phi^n}{n!} = \int_M (\alpha, d\tilde{f})_{g_\phi} \frac{\omega_\phi^n}{n!}$$

which we integrate by parts again to get

$$= \int_M \tilde{f} \cdot (\partial \bar{\partial} \phi_t * \partial \bar{\partial} \psi - \Delta_\phi \phi_t \cdot \Delta_\phi \psi) \frac{\omega_\phi^n}{n!},$$

where at the last equality we applied Lemma 3.3. From this we infer that

$$d^* \alpha = \partial \bar{\partial} \phi_t * \partial \bar{\partial} \psi - \Delta_\phi \phi_t \cdot \Delta_\phi \psi;$$

but, in view of (3.7) we deduce that $\partial \bar{\partial} \phi_t * \partial \bar{\partial} \psi - \Delta_\phi \phi_t \cdot \Delta_\phi \psi = \Delta_\phi f$, for some smooth real valued function f . Putting our latest equalities together, we get $d^* \alpha = d^* df$, that is $\pi_d(\alpha) = df$ and (3.8) is proved. This concludes the proof of the lemma. \square

We are now ready to present the covariant derivative for the Dirichlet metric in this environment.

Proposition 3.6. *Let $d\phi(t)$ be a smooth curve in the space \mathcal{A} and let $d\psi = d\psi(t)$ be a smooth curve of tangent vectors along the curve $d\phi$, with $d\psi(t) \in T_{d\phi(t)}\mathcal{A}$ for any $t \in [0, 1]$; then the formula*

$$(3.9) \quad D_t d\psi = d \frac{\partial \psi}{\partial t} + \frac{1}{2} \pi_d (C[\phi_t] * d\psi)$$

gives the Levi-Civita covariant derivative for the Dirichlet metric on the space \mathcal{A} .

Proof. First of all, let us notice that the right hand side of (3.9) is an exact one form; that is the operator D_t takes values in $T_{d\phi}\mathcal{A}$, as it must be.

About the metric compatibility (for its very definition, see [8, Definition 7, (1c)]), we compute

$$\begin{aligned} \frac{\partial}{\partial t} \langle d\psi, d\chi \rangle_{d\phi} &= \frac{\partial}{\partial t} \int_M (d\psi, d\chi)_{g_\phi} \frac{\omega_\phi^n}{n!} \\ &= - \int_M [(\partial \bar{\partial} \phi_t * \partial \psi, \bar{\partial} \chi)_{g_\phi} + (\partial \bar{\partial} \phi_t * \partial \chi, \bar{\partial} \psi)_{g_\phi}] \frac{\omega_\phi^n}{n!} \\ &\quad + \int_M [(d\psi_t, \chi)_{g_\phi} + (d\chi_t, \psi)_{g_\phi}] \frac{\omega_\phi^n}{n!} \\ &\quad + \int_M (d\psi, d\chi)_{g_\phi} \left(\Delta_\phi \frac{\partial \phi}{\partial t} \right) \frac{\omega_\phi^n}{n!}. \end{aligned} \quad (3.10)$$

Inserting the equation (3.9) into the equation (3.10) we have, by means of (3.8), the compatibility, i.e.

$$\frac{\partial}{\partial t} \langle d\psi, d\chi \rangle_{g_\phi} = \langle D_t d\psi, d\chi \rangle_{g_\phi} + \langle d\psi, D_t d\chi \rangle_{g_\phi}.$$

In order to show that the expression (3.9) is torsion free (see [8, Definition 7, (1d)]), let $d\phi(s, t)$ be a smooth two-parameter family of curves on \mathcal{A} , and let us compute

$$\begin{aligned} D_t d\phi_s - D_s d\phi_t &= \\ &= \frac{1}{2} \pi_d (C[\phi_t] * d\phi_s - C[\phi_s] * d\phi_t). \end{aligned} \quad (3.11)$$

Now we claim that, for any $d\tilde{f} \in T_{d\phi} \mathcal{A}$, we have

$$(3.12) \quad \langle D_t d\phi_s - D_s d\phi_t, d\tilde{f} \rangle_{g_\phi} = 0.$$

Indeed we have

$$\begin{aligned} 2 \langle D_t d\phi_s - D_s d\phi_t, d\tilde{f} \rangle_{g_\phi} &= \\ &= \int_M \left(\frac{1}{2} [C[\phi_t] * d\phi_s - C[\phi_s] * d\phi_t], d\tilde{f} \right)_{g_\phi} \frac{\omega_\phi^n}{n!} \\ &= - \int_M \left\{ (\partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_s + \Delta_\phi \phi_t \Delta_\phi \phi_s) \tilde{f} - (\partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_s + \Delta_\phi \phi_t \Delta_\phi \phi_s) \tilde{f} \right\} \frac{\omega_\phi^n}{n!} = 0, \end{aligned}$$

where at the first equality we used the fact that the projections (3.6) are orthogonal with respect to the Dirichlet metric; while at the last equality we integrated by parts and we applied Lemma 3.3. Thereby, claim (3.12) is proved and the expression (3.9) is the Levi-Civita covariant derivative for the space \mathcal{A} endowed with the Dirichlet metric. \square

In the following discussion we introduce the notation $D_t \phi_s$ which closely resembles the already defined element $D_t d\phi_s$. In fact, letting ϕ_s and $d\phi_s$ be corresponding elements via the map Φ_* defined in Lemma 3.1, it turns out that $D_t \phi_s$ which we are going to define corresponds to the already defined $D_t d\phi_s$. This will be useful in Remark 3.11.

Remark 3.7. *We notice that, with the notation of Definition 3.3, we have the following nice formula*

$$\frac{\partial}{\partial t} \langle d\psi, d\chi \rangle_{g_\phi} = \int_M \left((C[\phi_t] * d\psi, d\chi)_{g_\phi} + (d\frac{\partial \psi}{\partial t}, d\chi)_{g_\phi} + (d\psi, d\frac{\partial \chi}{\partial t})_{g_\phi} \right) \frac{\omega_\phi^n}{n!}.$$

Remark 3.8. *In the notation of the space \mathcal{H} , the Dirichlet metric is the pairing*

$$\langle \chi, \psi \rangle_\phi = \frac{1}{2} \int_M (d\chi, d\psi)_{g_\phi} \frac{\omega_\phi^n}{n!},$$

where $\psi, \chi \in T_\phi \mathcal{H}$. By means of the uniqueness of the Levi-Civita covariant derivative, which still holds in this infinite dimensional environment (see [21]), our expression (3.9) is equivalent to what we had found in [8], that is

$$(3.13) \quad \Delta_\phi D_t \psi = \Delta_\phi \frac{\partial \psi}{\partial t} + \left(\Delta_\phi \frac{\partial \phi}{\partial t} \right) (\Delta_\phi \psi) + \frac{\partial (\Delta_\phi \psi)}{\partial t}.$$

3.2. Sectional curvature. In the present subsection we are going to prove that the sectional curvature of the Dirichlet metric for a two plane spanned by tangent vectors $\phi_s, \phi_t \in T_\phi \mathcal{H}$ is bounded above and below by a constant K which depends only on the point ϕ and on one generator ϕ_s . This led us to conjecture that unlike the case when M is a Riemann surface, the sectional curvature is not flat.

The main result of the section will be the following.

Theorem 3.9. *If $d\phi(s, t)$ a smooth two parameter family of curves in \mathcal{A} such that $d\phi_s, d\phi_t$ are \mathbb{R} -linearly independent at $d\phi$, than there exists a positive constant $K = K(\omega_\phi, d\phi_s)$, such that we have the following bounds for the sectional curvature of the Dirichlet metric*

$$-K(\omega_\phi, d\phi_s) \leq K_{\text{Dirichlet}}(d\phi_s, d\phi_t)_\phi \leq K(\omega_\phi, d\phi_s).$$

In what follows we simplify further the notation, writing Δ for Δ_ϕ ;

Definition 3.6. Given $d\phi = d\phi(s, t)$ a smooth two-parameter family in \mathcal{A} , we call $A = A(s, t)$ and $B = B(s, t)$ the following real valued smooth functions on M .

$$A(s, t) := D_t \phi_s;$$

$$B(s, t) := \Delta A(s, t) = 2\Delta \phi_{st} + \Delta \phi_t \Delta \phi_s - \partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_s.$$

Remember that $d\phi_s \in T_{d\phi} \mathcal{A}$ and $\phi_s \in T_\phi \mathcal{H}$ corresponds via the map Φ_* of Lemma 3.1. Notice that A and B are symmetric with respect to the real parameters s and t .

Lemma 3.10. *Let $d\phi = d\phi(s, t, \sigma, \tau)$ be a four parameter family of curves in the space of Kähler metrics \mathcal{A} . Then, with the above notation we have*

$$\int_M D_\sigma D_\tau \phi_s \Delta \phi_t \frac{\omega_\phi^n}{n!} = \int_M \phi_t [2B_\sigma(s, \tau) + \Delta \phi_\sigma B(s, \tau) + \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} A(s, \tau)] \frac{\omega_\phi^n}{n!}. \quad (3.14)$$

Proof. We compute, with an integration by parts

$$\begin{aligned} & \int_M D_\sigma D_\tau \phi_s \Delta \phi_t \frac{\omega_\phi^n}{n!} \\ &= \int_M \phi_t \cdot \Delta(D_\sigma D_\tau \phi_s) \frac{\omega_\phi^n}{n!} \\ &= \int_M \phi_t \cdot \left[2 \frac{\partial}{\partial \sigma} \Delta D_\tau \phi_s + \Delta \phi_\sigma \Delta D_\tau \phi_s + \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} D_\tau \phi_s \right] \frac{\omega_\phi^n}{n!}; \end{aligned} \quad (3.15)$$

the latter expression of (3.15) is precisely the right hand side of the claimed (3.14), and thus the lemma is proved. \square

Before moving further, we fix another piece of notation.

Definition 3.7. Let $\phi = \phi(s, t) : [0, 1] \times [0, 1] \rightarrow \mathcal{H}$, a smooth two-parameter family of curves in \mathcal{H} . Define, for any fixed $s, t \in [0, 1]$, $a(s, t) \in C^\infty(M, \mathbb{R})$ to be

$$a(s, t) := A(s, t) - 2\phi_{st}. \quad (3.16)$$

Remark 3.11. For any fixed pair $s, t \in [0, 1]$, then the function $a(s, t)$ is actually an element of $T_{\phi(s, t)}\mathcal{H}$. Indeed, what we have to check is

$$\int_M a(s, t) \frac{\omega_\phi^n}{n!} = 0.$$

On the first addendum of $a(s, t)$, we have that

$$\int_M D_s \phi_t \frac{\omega_\phi^n}{n!} = 0,$$

by the very definition of covariant derivative. Also, on the second term of $a(s, t)$ we have

$$\int_M -2\phi_{st} \frac{\omega_\phi^n}{n!} = \frac{\partial}{\partial s} \int_M -2\phi_t \frac{\omega_\phi^n}{n!} + 2 \int_M \phi_t \Delta \phi_s \frac{\omega_\phi^n}{n!}.$$

The latter vanishes if and only if ϕ_s and ϕ_t are orthogonal and in this case, $a(s, t)$ is an element of $T_{\phi(s, t)}\mathcal{H}$.

Lemma 3.12. With the notation as in (3.16), for a two parameter family of curves $d\phi = d\phi(s, t)$ in \mathcal{A} , we have the formula

$$\begin{aligned} & \int_M \{D_s D_t \phi_s - D_t D_s \phi_s\} \Delta \phi_t \frac{\omega_\phi^n}{n!} = \\ &= \int_M \phi_t \{ \partial \bar{\partial} \phi_s * \partial \bar{\partial} a(s, t) - \Delta \phi_s \Delta a(s, t) - \partial \bar{\partial} \phi_t * \partial \bar{\partial} a(s, s) + \Delta \phi_t \Delta a(s, s) \} \frac{\omega_\phi^n}{n!}. \end{aligned} \quad (3.17)$$

Proof. We write more explicitly the formula (3.14), substituting the definition of $B(\tau)$; what we get is

$$\begin{aligned} & \int_M D_\sigma D_\tau \phi_s \Delta \phi_t \frac{\omega_\phi^n}{n!} = \\ &= \int_M \phi_t \{ 2B_\sigma(s, \tau) + \Delta \phi_\sigma \cdot B(s, \tau) + \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} A(s, \tau) \} \frac{\omega_\phi^n}{n!} = \\ &= \int_M \phi_t \{ 4\Delta \phi_{s\tau\sigma} - 4\partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} \phi_{s\tau} - 2\partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} \phi_\tau \cdot \Delta \phi_s + 2\Delta \phi_{\tau\sigma} \Delta \phi_s \\ &\quad - 2\partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} \phi_s \cdot \Delta \phi_\tau + 2\Delta \phi_{s\sigma} \Delta \phi_\tau - 2\partial \bar{\partial} \phi_{\sigma\tau} * \partial \bar{\partial} \phi_s + 2\partial \bar{\partial} \phi_\tau * \partial \bar{\partial} \phi_{s\sigma} \\ &\quad + 2\Delta \phi_\sigma \Delta \phi_{s\tau} + \Delta \phi_\sigma \Delta \phi_\tau \Delta \phi_s - \Delta \phi_\sigma \partial \bar{\partial} \phi_\tau * \partial \bar{\partial} \phi_s + \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} A(s, \tau) \} \frac{\omega_\phi^n}{n!}. \end{aligned} \quad (3.18)$$

Thus, using (3.18), by the mutual cancellation of the terms symmetric in σ and τ , we get

$$\begin{aligned}
& \int_M \{D_\sigma D_\tau \phi_s - D_\tau D_\sigma \phi_s\} \Delta \phi_t \frac{\omega_\phi^n}{n!} = \\
& \int_M \phi_t \{ \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} (A(s, \tau) - 2\phi_{s\tau}) - \Delta \phi_\tau \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} \phi_s \\
& \quad - \partial \bar{\partial} \phi_\tau * \partial \bar{\partial} (A(s, \sigma) - 2\phi_{s\sigma}) + \Delta \phi_\sigma \partial \bar{\partial} \phi_\tau * \partial \bar{\partial} \phi_s \} \frac{\omega_\phi^n}{n!} \\
& = \int_M \phi_t \{ \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} a(s, \tau) - \Delta \phi_\tau \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} \phi_s \\
& \quad - \partial \bar{\partial} \phi_\tau * \partial \bar{\partial} a(s, \sigma) + \Delta \phi_\sigma \partial \bar{\partial} \phi_\tau * \partial \bar{\partial} \phi_s \} \frac{\omega_\phi^n}{n!}
\end{aligned}
\tag{3.19}$$

From (3.19) we immediately obtain

$$\begin{aligned}
& \int_M \{D_s D_t \phi_s - D_t D_s \phi_s\} \Delta \phi_t \frac{\omega_\phi^n}{n!} = \\
& \int_M \phi_t \{ \partial \bar{\partial} \phi_s * \partial \bar{\partial} a(s, t) - \Delta \phi_t \partial \bar{\partial} \phi_s * \partial \bar{\partial} \phi_s - \partial \bar{\partial} \phi_t * \partial \bar{\partial} a(s, s) + \Delta \phi_s \partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_s \} \frac{\omega_\phi^n}{n!}.
\end{aligned}
\tag{3.20}$$

Now, it is worth noticing that

$$(3.21) \quad -\Delta \phi_s \partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_s + \Delta \phi_t \partial \bar{\partial} \phi_s * \partial \bar{\partial} \phi_s = \Delta \phi_s \Delta a(s, t) - \Delta \phi_t \Delta a(s, s);$$

the latter claim is obvious by taking the difference of

$$\Delta \phi_s \Delta a(s, t) = (\Delta \phi_s)^2 \Delta \phi_t - \Delta \phi_s \partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_s$$

and

$$\Delta \phi_t \Delta a(s, s) = \Delta \phi_t (\Delta \phi_s)^2 - \Delta \phi_t \partial \bar{\partial} \phi_s * \partial \bar{\partial} \phi_s.$$

Thus, combining (3.20) and (3.21) we get

$$\begin{aligned}
& \int_M \{D_s D_t \phi_s - D_t D_s \phi_s\} \Delta \phi_t \frac{\omega_\phi^n}{n!} = \\
& = \int_M \phi_t \{ \partial \bar{\partial} \phi_s * \partial \bar{\partial} a(s, t) - \Delta \phi_s \Delta a(s, t) - \partial \bar{\partial} \phi_t * \partial \bar{\partial} a(s, s) + \Delta \phi_t \Delta a(s, s) \} \frac{\omega_\phi^n}{n!},
\end{aligned}$$

which is precisely the claimed formula (3.17). This concludes the proof of the lemma. \square

Remark 3.13. *In the case of a Riemannian surface, for any two functions χ and ψ , we have that*

$$\Delta \psi \Delta \chi = \partial \bar{\partial} \psi * \partial \bar{\partial} \chi.$$

When we insert this fact in (3.17) we find back the conclusion, already got in [8], that the space of Kähler potentials endowed with the Dirichlet metric is flat.

A consequence of formula (3.17) is the following nice expression for the numerator of the sectional curvature of the Dirichlet metric.

Lemma 3.14. *The following formula holds for a smooth two parameter family of curves $d\phi = d\phi(s, t)$ in the space of Kähler metrics \mathcal{A} ;*

$$\begin{aligned} & \int_M \{D_s D_t \phi_s - D_t D_s \phi_s\} \Delta_\phi \phi_t \frac{\omega_\phi^n}{n!} = \\ &= \frac{1}{2} \int_M |da(s, t)|_{g_\phi}^2 \frac{\omega_\phi^n}{n!} - \frac{1}{2} \int_M (da(s, s), da(t, t))_{g_\phi} \frac{\omega_\phi^n}{n!}, \end{aligned} \quad (3.22)$$

where we recall that the symmetric expression $a(\sigma, \tau)$ satisfies

$$(3.23) \quad \Delta_\phi a(\sigma, \tau) = \Delta_\phi \phi_\sigma \Delta_\phi \phi_\tau - \partial \bar{\partial} \phi_\sigma * \partial \bar{\partial} \phi_\tau.$$

Proof. In the argument here, we are going to write Δ, g, ω instead of $\Delta_\phi, g_\phi, \omega_\phi$. We first consider this part of the right hand side of (3.17);

$$\int_M (\Delta a(s, s) \Delta \phi_t \cdot \phi_t - \partial \bar{\partial} a(s, s) * \partial \bar{\partial} \phi_t \cdot \phi_t) \frac{\omega^n}{n!},$$

which we integrate twice by parts using formula (3.4), getting

$$\begin{aligned} &= \int_M \left(-\frac{1}{2} (da(s, s), d\Delta \phi_t)_g \cdot \phi_t - \frac{1}{2} (da(s, s), d\phi_t)_g \cdot \Delta \phi_t \right. \\ &\quad \left. + \frac{1}{2} (da(s, s), d\Delta \phi_t)_g \cdot \phi_t + \frac{1}{2} (da(s, s), \partial \bar{\partial} \phi_t * d\phi_t)_g \right) \frac{\omega^n}{n!} \\ &= \int_M \left(\frac{1}{2} a(s, s) \cdot (d\phi_t, d\Delta \phi_t)_g + a(s, s) (\Delta^2 \phi_t) \cdot \phi_t \right. \\ &\quad + \frac{1}{2} a(s, s) (\Delta \phi_t)^2 + \frac{1}{2} a(s, s) (d\Delta \phi_t, d\phi_t)_g \\ &\quad - a(s, s) (\Delta^2 \phi_t) \cdot \phi_t - a(s, s) (d\phi_t, d\Delta \phi_t)_g \\ &\quad \left. - \frac{1}{2} a(s, s) (d\Delta \phi_t, d\phi_t)_g - a(s, s) \partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_t \right) \frac{\omega^n}{n!} \end{aligned}$$

and so, after obvious cancellations, we get

$$= \int_M (a(s, s) (\Delta \phi_t)^2 - a(s, s) \partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_t) \frac{\omega^n}{n!}$$

and finally, by the very definition of $a(t, t)$, we conclude

$$(3.24) \quad = \int_M a(s, s) \Delta a(t, t) \frac{\omega^n}{n!} = -\frac{1}{2} \int_M (da(s, s), da(t, t))_g \frac{\omega^n}{n!}.$$

Concerning the other part of the right hand side of (3.17), we similarly compute

$$\begin{aligned} & \int_M (-\Delta a(s, t) \Delta \phi_s \cdot \phi_t + \partial \bar{\partial} a(s, t) * \partial \bar{\partial} \phi_s \cdot \phi_t) \frac{\omega^n}{n!} \\ &= \int_M \left(\frac{1}{2} (da(s, t), d\Delta \phi_s)_g \cdot \phi_t + \frac{1}{2} (da(s, t), d\phi_t)_g \Delta \phi_s \right. \\ &\quad \left. - \frac{1}{2} (da(s, t), d\Delta \phi_s)_g \cdot \phi_t - \frac{1}{2} (da(s, t), \partial \bar{\partial} \phi_s * d\phi_t)_g \right) \frac{\omega^n}{n!} \end{aligned}$$

and again by parts, we obtain

$$\begin{aligned}
&= \int_M \left(-a(s, t)(\Delta^2 \phi_s) \cdot \phi_t - \frac{1}{2}a(s, t)(d\Delta \phi_s, d\phi_t)_g \right. \\
&\quad - a(s, t)\Delta \phi_t \Delta \phi_s - \frac{1}{2}a(s, t)(d\phi_t, d\Delta \phi_s)_g \\
&\quad + a(s, t)(\Delta^2 \phi_s) \cdot \phi_t + \frac{1}{2}a(s, t)(d\Delta \phi_s, d\phi_t)_g \\
&\quad \left. + \frac{1}{2}a(s, t)(d\Delta \phi_s, d\phi_t)_g + a(s, t)\partial\bar{\partial}\phi_s * \partial\bar{\partial}\phi_t \right) \frac{\omega^n}{n!}
\end{aligned}$$

and so we conclude

$$\begin{aligned}
&= \int_M (-a(s, t)(\Delta \phi_s)^2 + a(s, t)\partial\bar{\partial}\phi_s * \partial\bar{\partial}\phi_t) \frac{\omega_\phi^n}{n!} \\
&= \int_M -a(s, t)\Delta a(s, t) \frac{\omega_\phi^n}{n!} = \frac{1}{2} \int_M (da(s, t), da(s, t))_{g_\phi} \frac{\omega_\phi^n}{n!}.
\end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25) we get the claimed formula (3.22). \square

We are now going to get another equivalent description of the same quantity already considered in the previous lemma.

Lemma 3.15. *We have for a two parameter family of curves $d\phi = d\phi(s, t)$ in the space of Kähler metrics \mathcal{A} , the formula (which makes use of the piece of notation introduced in Definition 3.3 and in (3.4))*

$$\begin{aligned}
&\int_M \{D_s D_t \phi_s - D_t D_s \phi_s\} \Delta \phi_t \frac{\omega_\phi^n}{n!} \\
&= \int_M (C[a(s, t)] * \partial \phi_s, \bar{\partial} \phi_t)_{g_\phi} \frac{\omega_\phi^n}{n!} - \int_M (C[a(s, t)] * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi} \frac{\omega_\phi^n}{n!}.
\end{aligned} \tag{3.26}$$

Proof. Plug in the formula (3.17) the definition of $C[a(\cdot, \cdot)]$, and then integrate by parts. The equation (3.26) will be gotten after applying Lemma 3.3. \square

In what follows now, our aim is to further simplify the formula (3.26) without loss of generality.

Lemma 3.16. *The following formula holds*

$$\left| \int_M (C[a(s, t)] * \partial \phi_s, \bar{\partial} \phi_t)_{g_\phi} \frac{\omega_\phi^n}{n!} \right| \leq K_1(\phi_s) \int_M |\partial \phi_t|_{g_\phi}^2 \frac{\omega_\phi^n}{n!}, \tag{3.27}$$

where K_1 is a positive constant, depending on the tensor $C[\phi_s]$.

Proof. We notice that, using integration by parts and Lemma 3.3, we have

$$\int_M (C[a(s, t)] * \partial \phi_s, \bar{\partial} \phi_t)_{g_\phi} \frac{\omega_\phi^n}{n!} = - \int_M (C[a(s, t)] * \partial \bar{\partial} \phi_s) \cdot \phi_t \frac{\omega_\phi^n}{n!}. \tag{3.28}$$

Next, we remark that

$$(3.29) \quad C[a(s, t)] * \partial \bar{\partial} \phi_s = C[\phi_s] * \partial \bar{\partial} a(s, t);$$

indeed, by definition

$$C[a(s, t)] * \partial \bar{\partial} \phi_s = \Delta a(s, t) \Delta \phi_s - \partial \bar{\partial} a(s, t) * \partial \bar{\partial} \phi_s,$$

and by its symmetric structure we obtain (3.29). Also, we have

$$(3.30) \quad \left| \int_M (C[\phi_s] * \partial \bar{\partial} a(s, t)) \cdot \phi_t \frac{\omega_\phi^n}{n!} \right| \leq K_3(C[\phi_s]) \|\partial a(s, t)\| \|\partial \phi_t\|.$$

Then,

$$\begin{aligned} \left| \int_M a(s, t) \Delta_\phi a(s, t) \frac{\omega_\phi^n}{n!} \right| &= \left| \int_M (C[\phi_s] * \partial \phi_t, \bar{\partial} a(s, t))_{g_\phi} \frac{\omega_\phi^n}{n!} \right| \\ &\leq \epsilon \int_M |\partial a(s, t)|_{g_\phi}^2 \frac{\omega_\phi^n}{n!} + K_4(\epsilon, C[\phi_s]) \int_M |\partial \phi_t|_{g_\phi}^2 \frac{\omega_\phi^n}{n!}. \end{aligned}$$

Thus, choosing $\epsilon = \frac{1}{2}$ we have

$$(3.31) \quad \int_M |\partial a(s, t)|_{g_\phi}^2 \frac{\omega_\phi^n}{n!} \leq K_5(\phi_s) \int_M |\partial \phi_t|_{g_\phi}^2 \frac{\omega_\phi^n}{n!}.$$

Thus, the combination of equations (3.30) and (3.31) gives the claimed formula (3.27), whence the proof is complete. \square

Lemma 3.17. *The following formula, concerning the second term appearing on the right hand side of (3.26), holds*

$$(3.32) \quad \left| \int_M (C[a(s, s)] * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi} \frac{\omega_\phi^n}{n!} \right| \leq K_2(\phi_s) \int_M |\partial \phi_t|_{g_\phi}^2 \frac{\omega_\phi^n}{n!}.$$

Proof. We immediately estimate

$$\left| \int_M (C[a(s, s)] * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi} \frac{\omega_\phi^n}{n!} \right| \leq K_6(C[a(s, s)]) \int_M |\partial \phi_t|_{g_\phi}^2 \frac{\omega_\phi^n}{n!}.$$

Then, as $a(s, s)$ only depends on ϕ_s , we get that the constant $K_6(C[a(s, s)])$ is equal to some $K_2(\phi_s)$ and the proof is achieved. \square

We are now in position to prove the main result of the section

Proof of Theorem 3.9. As we want to compute the sectional curvature of a two plane, we have the freedom to replace the two sections $d\phi_s$ and $d\phi_t$ by two other sections which span the same plane and which are orthonormal. The latter is achieved simply by employing the Gram-Schmidt method. We still call the two orthogonal sections $d\phi_s$ and $d\phi_t$. Thus, the sectional curvature of the plane spanned by $d\phi_s$ and $d\phi_t$ is

$$K_D(d\phi_s, d\phi_t) = \int_M \{D_s D_t \phi_s - D_t D_s \phi_s\} \Delta \phi_t \frac{\omega_\phi^n}{n!}$$

Formula (3.26), combined with formulas (3.27) and (3.32), gives

$$\begin{aligned} & \left| \int_M \{D_s D_t \phi_s - D_t D_s \phi_s\} \Delta \phi_t \frac{\omega_\phi^n}{n!} \right| \\ & \leq (K_1(\phi_s) + K_2(\phi_s)) \cdot \int_M |\partial \phi_t|_{g_\phi}^2 \frac{\omega_\phi^n}{n!} = K(\phi_s), \end{aligned} \quad (3.33)$$

where we used the orthonormality assumption. This completes the proof. \square

Remark 3.18. *In view of formula 3.22, we notice that whenever at some point $\phi \in \mathcal{H}$ there are two \mathbb{R} -linearly independent tangent vectors ϕ_s, ϕ_t such that $a(s, s)$ or $a(t, t)$ is constant, then $K_D(\phi_s, \phi_t)_\phi$ is bigger or equal to zero. On the opposite side, whenever $a(s, t)$ is constant, then $K_D(\phi_s, \phi_t)_\phi$ is less or equal to zero. This fact suggests that the Dirichlet metric has curvature whose sign changes. If this conjecture were confirmed, the consequence would be that the Dirichlet metric behaves differently both from the Mabuchi metric, whose sectional curvature is non positive, and from the Calabi metric, whose sectional curvature is positive.*

Remark 3.19. *We notice here that the formula (3.16) resembles but is slightly different from the complex Hessian equation; a difference is that ϕ_s and ϕ_t are not convex as in the environment of that equation. As pointed out in the previous remark, solving the modified complex Hessian equations $a(s, s) = 0$, $a(s, t) = 0$ or $a(t, t) = 0$ (see (3.23)) would detect the sign of the sectional curvature of the Dirichlet metric. This gives a motivation to approach the study of this new kind of partial differential equations.*

3.3. Geodesic equation. Geodesics in the Mabuchi metric environment enter as main characters the conjectures of Donaldson's program [21]; moreover, both for the Mabuchi and the Calabi metric, they are the key tools to show that the space \mathcal{H} is endowed with the structure of a metric space. In fact, in both cases it turns out that they are length minimizing. Let us begin with two definitions which are by now classical, also in this infinite dimensional environment.

Definition 3.8. Let $d\phi = d\phi(t)$ be a smooth path in \mathcal{A} , with $t \in [a, b]$. We define the energy of $d\phi$ as

$$Engy(d\phi) = \int_a^b \left\{ \int_M (d\phi_t, d\phi_t)_{g_\phi} \frac{\omega_\phi^n}{n!} \right\} dt.$$

Also, we define the length of the same arc $d\phi(t)$ as

$$Lgth(d\phi) = \int_a^b \left\{ \int_M (d\phi_t, d\phi_t)_{g_\phi} \frac{\omega_\phi^n}{n!} \right\}^{1/2} dt.$$

Remark 3.20. *As well as in the finite dimensional Riemannian geometry theory, there still holds that geodesic arcs can be defined equivalently both as extremal points of the Energy functional and as solutions of the equation $D_t d\phi_t = 0$. The argument goes precisely in the same vein, that is by analyzing the first variation formula (cf. [26]).*

Definition 3.9. A bijective map $\lambda : \mathcal{A} \rightarrow \mathcal{A}$ is called an isometry of \mathcal{A} if, for every smooth path $d\phi(t)$ in \mathcal{A} , then its image $\lambda \circ d\phi(t)$ is still a smooth path, and moreover

$$Lgth(\lambda \circ d\phi) = Lgth(d\phi).$$

Proposition 3.21. Let $h \in \text{Aut}^0(M)$ be any holomorphic automorphism of M in the connected component of the identity. Then, the mapping $h^* : \mathcal{A} \rightarrow \mathcal{A}$, such that $d\phi \mapsto h^*(d\phi)$ is an isometry of \mathcal{A} endowed with the Dirichlet metric.

Proof. Consider a smooth arc $d\phi = d\phi(t)$ in \mathcal{A} , with $t \in [a, b]$. Denote by $d\chi$ the element of \mathcal{A} such that $h^*\omega = \omega + i\partial\bar{\partial}\chi$. Define $d\xi := d\chi + h^*d\phi$. Then,

$$\omega + i\partial\bar{\partial}\xi = \omega + i\partial(d\chi + h^*d\phi) = h^*(\omega + i\partial\bar{\partial}\phi) > 0,$$

which shows that $d\xi$ belongs to \mathcal{A} . Also, when we consider the time derivative, there holds $d\xi_t = h^*d\phi_t$; whence

$$\begin{aligned} Lgth(h^*d\phi) &= \int_a^b \left\{ \int_M (d\xi_t, d\xi_t)_{g_\xi} \frac{\omega_\xi^n}{n!} \right\}^{1/2} dt \\ &= \int_a^b \left\{ \int_M (h^*d\phi_t, h^*d\phi_t)_{g_\phi} h^* \left(\frac{\omega_\phi^n}{n!} \right) \right\}^{1/2} dt = Lgth(d\phi), \end{aligned}$$

which completes the proof of the proposition. \square

We are now going to prove that another result in [26] still holds in our setting, that is for the Dirichlet metric.

Definition 3.10. Let X be an holomorphic vector field on M ; $X \in H^0(M, \mathcal{O}(TM))$. We denote by $X_{\mathbb{R}}$ the real vector field defined by $X_{\mathbb{R}} := X + \overline{X}$. Also, we denote by $L_{X_{\mathbb{R}}}$ the Lie derivative of ω with respect to $X_{\mathbb{R}}$.

Proposition 3.22. Let X be an holomorphic vector field on M such that $L_{X_{\mathbb{R}}}(\omega) = 0$. Let be given a smooth path $d\phi = d\phi(t)$ in \mathcal{A} such that $L_{X_{\mathbb{R}}}(d\phi) = 0$ for all t . Then

$$L_{X_{\mathbb{R}}}(D_t d\psi_t) = D_t(L_{X_{\mathbb{R}}} d\psi),$$

for all the smooth sections $d\psi$ along the path $d\phi$.

Proof. Recall that formula (3.9) gives the following expression of the Levi-Civita covariant derivative

$$D_t d\psi = d\psi_t + \frac{1}{2} \pi_d (C[\phi_t] * d\psi) .$$

So, for $\frac{\partial}{\partial t}$ and π_d commute with D_t , it remains to check that

$$L_{X_{\mathbb{R}}} C[\phi_t] * \partial\psi = C[\phi_t] * (\partial(X_{\mathbb{R}}\psi)).$$

We claim that $L_{X_{\mathbb{R}}}C[\phi_t] = 0$; once achieved the claim, the above formula follows clearly. Thus, we compute, at a point p of a local coordinate chart (U, z^1, \dots, z^n) ,

$$\begin{aligned} L_{X_{\mathbb{R}}}C[\phi_t] &= -L_{X_{\mathbb{R}}}\partial\bar{\partial}\phi_t + L_{X_{\mathbb{R}}}(\Delta_{\phi}\phi_t)\omega \\ &= -\frac{\partial}{\partial t}\partial\bar{\partial}(\mathbb{R}\phi) + L_{X_{\mathbb{R}}}\left(\frac{\partial}{\partial t}(\Delta_{\phi}\phi_t + \phi_t^{i\bar{j}}\phi_{ti\bar{j}})\right)\omega \\ &= \frac{\partial}{\partial t}(X_{\mathbb{R}}(\Delta_{\phi}\phi_t) + \frac{1}{2}L_{X_{\mathbb{R}}}(\phi_t^{i\bar{j}}\phi_{ti\bar{j}}))\omega = 0, \end{aligned}$$

where we used repeatedly the assumptions $X_{\mathbb{R}}\phi = 0$ and $L_{X_{\mathbb{R}}}(\omega) = 0$. Note that the argument for proving that $L_{X_{\mathbb{R}}}C[\phi_t] * \partial\psi = C[\phi_t] * (\partial(X_{\mathbb{R}}\psi))$ goes exactly the same way. Whence, the proof of the proposition is complete. \square

Let us also recall other classical definitions, rephrased for the space \mathcal{A} , in view of the next proposition.

Definition 3.11. Let $d\phi = d\phi(t)$ be a smooth path in \mathcal{A} , with $t \in [0, 1]$. We denote by $S = S(\phi)$ the scalar curvature of the metric ω_{ϕ} . It is a very well known fact that the scalar $\underline{S} := \frac{1}{Vol} \int_M S(\phi) \frac{\omega_{\phi}^n}{n!}$ is a constant that only depends on the Kähler class $[\omega_{\phi}]$. We define $f := f(\phi)$ as the zero-mean function satisfying $\Delta_{\phi}f = S - \underline{S}$. Finally, the K -energy (or Mabuchi energy) is defined as (compare [25, Definition (3.1)])

$$\nu(\phi) := \frac{1}{2} \int_0^1 \int_M (d\phi_t, df)_{g_{\phi}} \frac{\omega_{\phi}^n}{n!} dt.$$

Proposition 3.23. *Let $d\phi = d\phi(t)$ be a geodesic for the Dirichlet metric, with $t \in [0, 1]$. Then, the K -energy is convex at a point $d\phi(t_0)$ which corresponds to a constant scalar curvature Kähler metric.*

Proof. By means of the metric compatibility of the Levi-Civita covariant derivative, we compute

$$\frac{\partial^2}{\partial t^2} \nu(\phi) = \frac{1}{2} \int_M [(D_t d\phi_t, df)_{g_{\phi}} + (d\phi_t, D_t df)_{g_{\phi}}] \frac{\omega_{\phi}^n}{n!}.$$

The first addendum vanishes since, by assumption on $d\phi$, there holds $D_t d\phi_t = 0$. So we reduced the question to the study of

$$I := \int_M (d\phi_t, D_t df)_{g_{\phi}} \frac{\omega_{\phi}^n}{n!}.$$

We preliminarily compute, using the very definition of D_t ,

$$\begin{aligned} I &= \int_M (d\phi_t, d\frac{\partial f}{\partial t} + i_{ReX}C[\phi_t])_{g_{\phi}} \frac{\omega_{\phi}^n}{n!} \\ &= \int_M \left[-\phi_t \Delta_{\phi} \frac{\partial f}{\partial t} + \frac{1}{4} (C[\phi_t] * d\phi_t, df)_{g_{\phi}} \right] \frac{\omega_{\phi}^n}{n!} \\ (3.34) \quad &= \int_M \left[-\phi_t \Delta_{\phi} \frac{\partial f}{\partial t} + \frac{1}{2} (\partial\bar{\partial}\phi_t * \partial\bar{\partial}\phi_t - (\Delta_{\phi}\phi_t)^2) f \right] \frac{\omega_{\phi}^n}{n!}. \end{aligned}$$

Our next goal is to analyze the addendum $\int_M -\phi_t \Delta_{\phi} \frac{\partial f}{\partial t} \frac{\omega_{\phi}^n}{n!}$. First, notice that, using the notation suggested in Definition 3.4, we can write the scalar curvature as $S = \omega_{\phi} * \rho_{\phi}$, where ρ_{ϕ} is the Ricci form corresponding to ω_0 . Time-differentiating

formula $S = \Delta_\phi f + \underline{S}$, we get $\Delta_\phi f_t = S_t + \partial \bar{\partial} \phi_t * \partial \bar{\partial} f$. Also, time-differentiating $S = \omega_\phi * \rho_\phi$, we get $S_t = -\Delta_\phi^2 \phi_t - \partial \bar{\partial} \phi_t * \rho_\phi$. Inserting the latter expression into the former one, we end up with

$$\Delta_\phi f_t = -\Delta_\phi^2 \phi_t - \partial \bar{\partial} \phi_t * (\rho_\phi - \partial \bar{\partial} f).$$

Thus, we can rewrite the term we are presently concerned about as

$$\begin{aligned} & \int_M -\phi_t \Delta_\phi f_t \frac{\omega_\phi^n}{n!} \\ &= \int_M -\phi_t \{-\Delta_\phi^2 \phi_t + \phi_t * (\partial \bar{\partial} f - \rho_\phi)\} \frac{\omega_\phi^n}{n!} \\ (3.35) \quad &= \int_M \{(\Delta_\phi \phi_t)^2 - ((\rho_\phi - \partial \bar{\partial} f) * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi}\} \frac{\omega_\phi^n}{n!}, \end{aligned}$$

where at the last equality we used the fact that the tensor $\rho_\phi - \partial \bar{\partial} f$ is divergence free; this can be seen, using the second Bianchi identity, in local coordinate chart (U, z^1, \dots, z^n) ; namely, $(\rho_{i\bar{j}} - f_{i\bar{j}})^i = S_{\bar{j}} - S_{\bar{j}} = 0$.

Now, our aim is to expand the second term of (3.34). We compute

$$\begin{aligned} & \frac{1}{2} \int_M (\partial \bar{\partial} \phi_t * \partial \bar{\partial} \phi_t - (\Delta_\phi \phi_t)^2) f \frac{\omega_\phi^n}{n!} = \frac{1}{2} \int_M [\partial \bar{\partial} \phi_t * \partial \bar{\partial} f - \Delta_\phi \phi_t \Delta_\phi f] \phi_t \frac{\omega_\phi^n}{n!} \\ (3.36) \quad &= \frac{1}{2} \int_M (C[f] * \partial \phi_t, \bar{\partial} \phi_t)_g \frac{\omega_\phi^n}{n!}, \end{aligned}$$

where the argument behind the first equality is exactly the same as in formula (3.24). We plug (3.35) and (3.36) in (3.34) and we get

$$\begin{aligned} I &= \int_M [(\Delta_\phi \phi_t)^2 - ((\rho_\phi - \partial \bar{\partial} f) * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi}] \frac{\omega_\phi^n}{n!} \\ &\quad + \frac{1}{2} \int_M [-(\partial \bar{\partial} f * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi} + (S - \underline{S})(\partial \phi_t, \bar{\partial} \phi_t)_{g_\phi}] \frac{\omega_\phi^n}{n!} \\ &= \int_M [(\Delta_\phi \phi_t)^2 - (\rho_\phi * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi}] \frac{\omega_\phi^n}{n!} \\ &\quad + \frac{1}{2} \int_M [(\partial \bar{\partial} f * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi} + (S - \underline{S})(\partial \phi_t, \bar{\partial} \phi_t)_{g_\phi}] \frac{\omega_\phi^n}{n!} \\ &= \frac{1}{2} \int_M \{2|D\phi_t|_{g_\phi}^2 + ([\partial \bar{\partial} f + (S - \underline{S})g_\phi] * \partial \phi_t, \bar{\partial} \phi_t)_{g_\phi}\} \frac{\omega_\phi^n}{n!}, \end{aligned}$$

where at the last equality we employed the Ricci identity and we wrote $D\phi_t$ which stands, in a coordinate chart, for $g^{\alpha\bar{\lambda}} \frac{\partial^2 \phi_t}{\partial z^\lambda \partial z^{\bar{\beta}}} \frac{\partial}{\partial z^\alpha} \otimes dz^{\bar{\beta}}$ (see [4, page 100, c)). In particular, the above formula shows that, if $d\phi(t_0)$ induces a constant scalar curvature Kähler metric, that is $S(\phi(t_0)) = \underline{S}$ and $f(\phi(t_0)) = 0$, then

$$\frac{\partial^2}{\partial t^2} \nu(\phi)(t_0) \geq 0,$$

which completes the proof of the proposition. \square

Remark 3.24. *It is pointed out by Xiu Xiong Chen and the second author (see [17], Remark 3.2) that the Pseudo-Calabi flow is the gradient flow of the K-energy for the Dirichlet metric. Here, we want to notice that also the (normalized) Kähler Ricci*

flow is the gradient flow of the K -energy in the first Chern class for the Dirichlet metric. Indeed, the first variation of the K energy is, just from the very definition,

$$\frac{\partial \nu}{\partial t}(d\phi) = \int_M (d\phi_t, df)_{g_\phi} \frac{\omega_\phi^n}{n!}.$$

Moreover the normalized Kähler Ricci flow at the potentials level is

$$\frac{\partial}{\partial t} \phi = \log \left(\frac{\omega_\phi^n}{\omega^n} \right) + \phi + h_\omega + c(t).$$

Then, taking $\partial\bar{\partial}$ on both sides and using the Maximum Principle, we get the claimed conclusion.

4. THE FAMILY OF WEIGHTED METRICS

Here we introduce a family of Riemannian structures, whose distinguished element is the Calabi metric.

Definition 4.1. We consider the following family of Riemannian structures on the space \mathcal{C} , which we label as *weighted metrics*; let be given $F \in \mathcal{C}$ and $G \in T_F \mathcal{C}$, and let $\chi \in C^\infty([0, +\infty[,]0, +\infty[)$. Then we define

$$(4.1) \quad \langle G_1, G_2 \rangle_{F; \chi} := \int_M \chi(F) G_1 G_2 \frac{\omega^n}{n!}.$$

We call the function χ the *weight* of the Riemannian product defined above. Any such Riemannian product is smooth (a role here is played by the smoothness of the weight); moreover, any such Riemannian product is non-degenerate (since the weight is a positive function).

Remark 4.1. The space of Kähler metrics \mathcal{H} together with the following Riemannian product is isometric to the space \mathcal{C} with the Riemannian product (4.1); given $\omega_\phi \in \mathcal{H}$, and $\psi \in T_{\omega_\phi} \mathcal{H}$, that product is given by

$$\langle \psi_1, \psi_2 \rangle_{\omega_\phi; \chi} = \int_M \chi \left(\frac{\omega_\phi^n}{\omega^n} \right) (\Delta_\phi \psi_1) \cdot (\Delta_\phi \psi_2) \left(\frac{\omega_\phi^n}{\omega^n} \right)^2 \frac{\omega^n}{n!}.$$

Remark 4.2. A special sub-family of Riemannian products is got by specifying the weight to be of the form $\chi(x) = x^k$, for some $k \in \mathbb{Z}$. In this case, we are considering the metric, for a fixed $k \in \mathbb{Z}$,

$$\langle G_1, G_2 \rangle_F := \int_M G_1 G_2 F^k \omega^n.$$

The Calabi metric is retrieved for $k = -1$: that is the weight that gives the Calabi metric is $\chi(x) = x^{-1}$. According to Remark 4.1, we also retrieve its expression at the level of the space of Kähler metrics, which is

$$\langle \psi_1, \psi_2 \rangle_{\omega_\phi; x^{-1}} = \int_M (\Delta_\phi \psi_1) \cdot (\Delta_\phi \psi_2) \frac{\omega_\phi^n}{n!},$$

(compare with the expression in [8]).

The following sections concerns the computation of formulas and equations that come from the definition of weighted metrics (4.1).

4.1. The Levi-Civita connection. We have the existence of the Levi-Civita covariant derivative for any of the Riemannian structures defined by (4.1). For the precise definition of the Levi-Civita covariant derivative, we refer to [8, Definition 7]

Proposition 4.3. *The Levi-Civita covariant derivative for the Riemannian structure (4.1) exists and is unique. Let $F = F(t)$ be a smooth path in \mathbb{C} and let $G = G(t)$ a vector field on the path; then the Levi-Civita covariant derivative of G along F is given by*

$$(4.2) \quad D_t^\chi G = \frac{dG}{dt} + \frac{\chi'(F)}{2\chi(F)} \frac{dF}{dt} G - \frac{\chi^{-1}(F)}{\int_M \chi^{-1}(F) \frac{\omega^n}{n!}} \int_M \frac{\chi'(F)}{2\chi(F)} \frac{dF}{dt} G \frac{\omega^n}{n!},$$

where the inverse of χ is understood with respect to the multiplicative structure, i.e. $\chi^{-1} = \frac{1}{\chi}$.

Proof. The additivity, the homogeneity and the Leibniz rule of (4.2) are evident. Concerning the compatibility with the metric, we compute

$$\begin{aligned} \frac{d}{dt} \langle G, G \rangle_{F;\chi} &= \frac{d}{dt} \int_M \chi(F) G^2 \frac{\omega^n}{n!} \\ &= 2 \int_M G \chi(F) \left(\frac{dG}{dt} + \frac{\chi'(F)}{2\chi(F)} \frac{dF}{dt} G \right) \frac{\omega^n}{n!}. \end{aligned}$$

The trick now is that we can add, inside the round brackets, a term of the form $\lambda \chi^{-1}(F)$, where λ is any real constant. This will not change the value of the integral, since we have the constraint $\int_M G \frac{\omega^n}{n!}$ on tangent vectors. Thus, we can write

$$\frac{d}{dt} \langle G, G \rangle_{F;\chi} = 2 \int_M G \chi(F) \left(\frac{dG}{dt} + \frac{\chi'(F)}{2\chi(F)} \frac{dF}{dt} G + \lambda \chi^{-1}(F) \right) \frac{\omega^n}{n!}.$$

Now, the constant λ is chosen in order to have a tangent vector insider the round brackets; namely, we want the following constraint to be fulfilled

$$\int_M \left(\frac{dG}{dt} + \frac{\chi'(F)}{2\chi(F)} \frac{dF}{dt} G + \lambda \chi^{-1}(F) \right) \frac{\omega^n}{n!} = 0.$$

The above equality holds if and only if λ is

$$\lambda = - \frac{1}{\int_M \chi^{-1}(F) \frac{\omega^n}{n!}} \int_M \frac{\chi'(F)}{2\chi(F)} \frac{dF}{dt} G \frac{\omega^n}{n!},$$

according to the statement. Finally, to verify the property on torsion free, we let $F = F(s, t)$ be a two parameter family of smooth curves, then we take $G = \frac{dF}{ds}$, and we notice that the equation (4.2) is symmetric in s and t . This completes the proof of the proposition. \square

4.2. The Sectional Curvature. We prepare the main result of this section via some preliminary lemmas. In this section, let $F = F(s, t)$ denote a two-parameter family of smooth curves on \mathbb{C} . Also, let $G = G(s, t), H = H(s, t) \in T_{F(s, t)} \mathbb{C}$ be smooth curves of tangent vectors.

Lemma 4.4. *The following formula holds*

$$\begin{aligned}
& D_s^\chi D_t^\chi G - D_t^\chi D_s^\chi G \\
&= \frac{1}{4 \int_M \chi^{-1} \frac{\omega^n}{n!}} \frac{\chi'(F)}{\chi^2(F)} F_s \int_M \frac{\chi'(F)}{\chi(F)} F_t G \frac{\omega^n}{n} - \frac{1}{4 \int_M \chi^{-1} \frac{\omega^n}{n!}} \frac{\chi'(F)}{\chi^2(F)} F_t \int_M \frac{\chi'(F)}{\chi(F)} F_s G \frac{\omega^n}{n} \\
&+ \frac{\chi^{-1}(F)}{2} \int_M \frac{\chi'(F)}{\chi^2(F)} F_s \frac{\omega^n}{n!} \int_M \frac{\chi'(F)}{\chi(F)} F_t G \frac{\omega^n}{n!} - \frac{\chi^{-1}(F)}{2} \int_M \frac{\chi'(F)}{\chi^2(F)} F_t \frac{\omega^n}{n!} \int_M \frac{\chi'(F)}{\chi(F)} F_s G \frac{\omega^n}{n!} \\
&+ \frac{\chi^{-1}(F)}{4} \left(\int_M \chi^{-1}(F) \frac{\omega^n}{n!} \right)^2 \int_M \frac{\chi'(F)}{\chi^2(F)} F_s \frac{\omega^n}{n!} \int_M \frac{\chi'(F)}{\chi(F)} F_t G \frac{\omega^n}{n!} \\
&- \frac{\chi^{-1}(F)}{4} \left(\int_M \chi^{-1}(F) \frac{\omega^n}{n!} \right)^2 \int_M \frac{\chi'(F)}{\chi^2(F)} F_t \frac{\omega^n}{n!} \int_M \frac{\chi'(F)}{\chi(F)} F_s G \frac{\omega^n}{n!}.
\end{aligned}$$

Proof. The claimed formula is lengthy computation which just makes use of (4.2). \square

Remark 4.5. *In the case of the Calabi metric, the last four addenda vanish. In fact, when $\chi(x) = x^{-1}$, $x > 0$, then the ratio $\frac{\chi'(F)}{\chi^2(F)} = -1$, and so the integral*

$$\int_M \frac{\chi'(F)}{\chi^2(F)} F_\bullet \frac{\omega^n}{n!},$$

appearing in all of the four addenda of the formula, vanishes since any tangent vector F_\bullet has to have mean zero by (2.5).

Lemma 4.6. *The following formula holds*

$$\begin{aligned}
& \langle D_s^\chi D_t^\chi G - D_t^\chi D_s^\chi G, H \rangle_{F; \chi} \\
&= \frac{1}{4 \int_M \chi^{-1} \frac{\omega^n}{n!}} \int_M \frac{\chi'(F)}{\chi(F)} F_s H \frac{\omega^n}{n} \int_M \frac{\chi'(F)}{\chi(F)} F_t G \frac{\omega^n}{n} \\
&- \frac{1}{4 \int_M \chi^{-1} \frac{\omega^n}{n!}} \int_M \frac{\chi'(F)}{\chi(F)} F_t H \frac{\omega^n}{n} \int_M \frac{\chi'(F)}{\chi(F)} F_s G \frac{\omega^n}{n}.
\end{aligned}$$

Proof. The formula follows from a straightforward computation that builds on the symmetries given by the Lemma 4.4. \square

We are now ready for the main result of this section

Theorem 4.7. *The sectional curvature of a two plane spanned by F_s and F_t has the following expression*

(4.3)

$$\begin{aligned}
& K^\chi[F](F_s, F_t) \\
&= \frac{1}{4 \int_M \chi^{-1} \frac{\omega^n}{n!}} \frac{\left(\int_M \frac{\chi'(F)}{\chi(F)} (F_s)^2 \frac{\omega^n}{n!} \right) \left(\int_M \frac{\chi'(F)}{\chi(F)} (F_t)^2 \frac{\omega^n}{n!} \right) - \left(\int_M \frac{\chi'(F)}{\chi(F)} F_s F_t \frac{\omega^n}{n!} \right)^2}{\left(\int_M \chi(F) (F_s)^2 \frac{\omega^n}{n!} \right) \left(\int_M \chi(F) (F_t)^2 \frac{\omega^n}{n!} \right) - \left(\int_M \chi(F) F_s F_t \frac{\omega^n}{n!} \right)^2}.
\end{aligned}$$

Proof. It follows directly from Lemma 4.6. \square

Corollary 4.8. *For every weight function χ , then the space \mathcal{C} always has non-negative sectional curvature. If χ is constant, then the right-hand side of (4.3) is clearly zero.*

Remark 4.9. *It can be immediately read off equation (4.3) that the Calabi metric gives a constant scalar curvature of value $\frac{1}{4V}$.*

Remark 4.10. *When the function χ^{-1} has convexity or concavity properties, then the Jensen inequality provides an estimate for the factor $\int_M \chi^{-1}(F) \frac{\omega^n}{n!}$.*

4.3. The Geodesic equation. The aim of the present subsection is to write down the geodesic equation for each one of the weighted metrics. We will solve the equation in a special case, and also we remark here that the Calabi metric case was already considered. In all the remainder cases, it would be interesting to know whether there are smooth solutions or not to the geodesic equation.

Proposition 4.11. *The geodesic equation is*

$$(4.4) \quad D_t^\chi \frac{dF}{dt} = \frac{d^2 F}{dt^2} + \frac{\chi'(F)}{2\chi(F)} \left(\frac{dF}{dt} \right)^2 - \frac{\chi^{-1}(F)}{\int_M \chi^{-1}(F) \frac{\omega^n}{n!}} \int_M \frac{\chi'(F)}{2\chi(F)} \left(\frac{dF}{dt} \right)^2 \frac{\omega^n}{n!} = 0.$$

Remark 4.12. *We would like to rewrite an equivalent form of (4.4) with respect to the Kähler potential ϕ . So, we multiply by a factor F^{-1} , and using that $F = \frac{\omega_\phi^n}{\omega^n}$, we have*

$$(4.5) \quad \begin{aligned} & \frac{d}{dt}(\Delta_\phi \dot{\phi}) + (\Delta_\phi \dot{\phi})^2 + \frac{\omega_\phi^n}{\omega^n} \frac{\chi'(\frac{\omega_\phi^n}{\omega^n})}{2\chi(\frac{\omega_\phi^n}{\omega^n})} (\Delta_\phi \dot{\phi})^2 \\ & - \frac{\omega_\phi^n}{\omega_\phi^n} \frac{\chi^{-1}(\frac{\omega_\phi^n}{\omega^n})}{\int_M \chi^{-1}(\frac{\omega_\phi^n}{\omega^n}) \frac{\omega_\phi^n}{n!}} \int_M \frac{\chi'(\frac{\omega_\phi^n}{\omega^n})}{2\chi(\frac{\omega_\phi^n}{\omega^n})} \left(\frac{\omega_\phi^n}{\omega^n} \Delta_\phi \dot{\phi} \right)^2 \frac{\omega^n}{n!} = 0. \end{aligned}$$

Remark 4.13. *The case of the Calabi metric can be retrieved in the following way; when $\chi(x) = x^{-1}$, then the ratio $\frac{\chi'(F)}{\chi(F)} = -F^{-1}$, so that the equation (4.4) reads*

$$(4.6) \quad \frac{d^2 F}{dt^2} - \frac{F^{-1}}{2} \left(\frac{dF}{dt} \right)^2 + \frac{F}{2V} \int_M \frac{1}{F} \left(\frac{dF}{dt} \right)^2 \frac{\omega^n}{n!} = 0.$$

Then, using that $2F \frac{d^2 F}{dt^2} - \left(\frac{dF}{dt} \right)^2 = 4F^{\frac{3}{2}} \frac{d^2 \sqrt{F}}{dt^2}$, we write the equation (4.6) as

$$\frac{d^2 \sqrt{F}}{dt^2} + \frac{\sqrt{F}}{4V} \int_M \frac{1}{F} \left(\frac{dF}{dt} \right)^2 \frac{\omega^n}{n!} = 0.$$

The key ingredient now is to recognize that the integral $\int_M \frac{1}{F} \left(\frac{dF}{dt} \right)^2 \frac{\omega^n}{n!}$ is a constant of motion. Then the Calabi geodesics are solutions of the harmonic motion equation, and they make sense as long as F is positive.

4.4. Convexity of the K -energy along geodesics. In this section we address the question on the convexity of the Mabuchi K -energy along the geodesics of some scalar product of the family (4.1), provided the existence of such geodesics. Thus, at the present level, the discussion is formal. By the way, we will compute an interesting example for which the computation will be substantial.

Proposition 4.14. *For a generic element of the family of scalar products (4.1), the second variation of the Mabuchi energy along its geodesic (provided its existence) is given by the following formula*

$$\begin{aligned} \frac{d^2\nu}{dt^2} &= \int_M (\Delta_\phi \phi')^2 \frac{\omega_\phi^n}{n!} \\ &+ \int_M \left[-\frac{\omega_\phi^n}{\omega_0^n} \frac{\chi'(\frac{\omega_\phi^n}{\omega_0^n})}{2\chi(\frac{\omega_\phi^n}{\omega_0^n})} (\Delta_\phi \phi')^2 + \frac{\omega_\phi^n}{\omega_0^n} \frac{\chi^{-1}(\frac{\omega_\phi^n}{\omega_0^n})}{\int_M \chi^{-1}(\frac{\omega_\phi^n}{\omega_0^n}) \frac{\omega_\phi^n}{n!}} \int_M \frac{\chi'(\frac{\omega_\phi^n}{\omega_0^n})}{2\chi(\frac{\omega_\phi^n}{\omega_0^n})} \left(\frac{\omega_\phi^n}{\omega_0^n} \Delta_\phi \phi' \right)^2 \frac{\omega_\phi^n}{n!} \right] \\ &\cdot \left(\log \frac{\omega_\phi^n}{\omega_0^n} - P \right) \frac{\omega_\phi^n}{n!} + \int_M ([\partial\bar{\partial}P - \rho(\omega_0)] * \partial\phi', \bar{\partial}\phi')_{g_\phi} \frac{\omega_\phi^n}{n!}. \end{aligned}$$

Here above $\Delta_\phi P := \omega_\phi * \rho(\omega_0) - \underline{S}$, and $\rho(\omega_0) = \rho(0)$ is the Ricci form corresponding to the Kähler form ω_ϕ .

Proof.

$$\begin{aligned} \frac{d^2\nu}{dt^2} &= -\frac{d}{dt} \int_M \phi' (S(\phi) - \underline{S}) \frac{\omega_\phi^n}{n!} = -\frac{d}{dt} \int_M \phi' \omega_\phi * \rho(\phi) \frac{\omega_\phi^n}{n!} \\ &= -\frac{d}{dt} \int_M \phi' \omega_\phi * (\rho(\phi) - \rho(0)) \frac{\omega_\phi^n}{n!} - \frac{d}{dt} \int_M \phi' \omega_\phi * \rho(0) \frac{\omega_\phi^n}{n!}. \end{aligned}$$

The first term is

$$\begin{aligned} &= \frac{d}{dt} \int_M \phi' \left(\Delta_\phi \log \frac{\omega_\phi^n}{\omega_0^n} \right) \frac{\omega_\phi^n}{n!} = \frac{d}{dt} \int_M \Delta_\phi \phi' \left(\log \frac{\omega_\phi^n}{\omega_0^n} \right) \frac{\omega_\phi^n}{n!} \\ &= \int_M \left(\frac{d}{dt} \Delta_\phi \phi' \right) \left(\log \frac{\omega_\phi^n}{\omega_0^n} \right) \frac{\omega_\phi^n}{n!} + \int_M (\Delta_\phi \phi')^2 \frac{\omega_\phi^n}{n!} + \int_M \Delta_\phi \phi' \left(\log \frac{\omega_\phi^n}{\omega_0^n} \right) (\Delta_\phi \phi') \frac{\omega_\phi^n}{n!} \\ &= \int_M (\Delta_\phi \phi')^2 \frac{\omega_\phi^n}{n!} + \int_M \left[\frac{d}{dt} (\Delta_\phi \phi') + (\Delta_\phi \phi')^2 \right] \left(\log \frac{\omega_\phi^n}{\omega_0^n} \right) \frac{\omega_\phi^n}{n!}. \end{aligned}$$

The second term is

$$\begin{aligned} &-\frac{d}{dt} \int_M \phi' \omega_\phi * \rho(0) \frac{\omega_\phi^n}{n!} = -\frac{d}{dt} \int_M \phi' [\omega_\phi * \rho(0) - \underline{S}] \frac{\omega_\phi^n}{n!} \\ &= -\frac{d}{dt} \int_M \phi' \Delta_\phi P \frac{\omega_\phi^n}{n!} = -\frac{d}{dt} \int_M (\Delta_\phi \phi') P \frac{\omega_\phi^n}{n!} \\ &= -\int_M \left[\frac{d}{dt} (\Delta_\phi \phi') + (\Delta_\phi \phi')^2 \right] P \frac{\omega_\phi^n}{n!} - \int_M \phi' \Delta_\phi P' \frac{\omega_\phi^n}{n!} \\ &= -\int_M \left[\frac{d}{dt} (\Delta_\phi \phi') + (\Delta_\phi \phi')^2 \right] P \frac{\omega_\phi^n}{n!} + \int_M ([\partial\bar{\partial}P - \rho(\omega_0)] * \partial\phi', \bar{\partial}\phi')_{g_\phi} \frac{\omega_\phi^n}{n!}. \end{aligned}$$

So, we found that

$$\begin{aligned} \frac{d^2\nu}{dt^2} &= \int_M (\Delta_\phi \phi')^2 \frac{\omega_\phi^n}{n!} \\ &+ \int_M \left[\frac{d}{dt} (\Delta_\phi \phi') + (\Delta_\phi \phi')^2 \right] \left(\log \frac{\omega_\phi^n}{\omega_0^n} - P \right) \frac{\omega_\phi^n}{n!} \\ &+ \int_M ([\partial\bar{\partial}P - \rho(\omega_0)] * \partial\phi', \bar{\partial}\phi')_{g_\phi} \frac{\omega_\phi^n}{n!}, \end{aligned}$$

and plugging in (4.5), we complete the proof of the proposition. \square

We want to give an example of a scalar product of the family (4.1) such that the Mabuchi K -energy is convex along its geodesic, at least in some special cases.

4.5. Entropy function. Recall the explicit form of K -energy in Chen [11] and Tian [28],

$$\begin{aligned} \nu_\omega(\varphi) &= \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n + \frac{S}{V} \sum_{i=0}^n \frac{n!}{(i+1)!(n-i)!} \int_M \varphi \omega^{n-i} \wedge (\partial \bar{\partial} \varphi)^i \\ &\quad - \frac{1}{V} \sum_{i=0}^{n-1} \frac{n!}{(i+1)!(n-i-1)!} \int_M \varphi Ric \wedge \omega^{n-1-i} \wedge (\partial \bar{\partial} \varphi)^i. \end{aligned}$$

Here, we consider the leading term of the K -energy.

Definition 4.2. We introduce the entropy function in the the space of volume conformal factors \mathcal{C} (see (2.1))

$$(4.7) \quad \mathcal{E}(F) = \int_M F \log(F) \frac{\omega^n}{n!}.$$

In which, F is the volume conformal factor $\frac{\omega_\varphi^n}{\omega^n}$.

We compute that along a differential curve $F(t)$,

$$\frac{d}{dt} \mathcal{E}(F(t)) = \int_M F' \log(F) \frac{\omega^n}{n!} + V.$$

Since the volume of the Kähler metrics in a fixed Kähler class is a topological constant, we obtain the following second order derivative

$$(4.8) \quad \frac{d^2}{dt^2} \mathcal{E}(F(t)) = \frac{d}{dt} \int_M F' \log(F) \frac{\omega^n}{n!}.$$

Now assuming that $F(t)$ is a geodesic and using (4.4), we get

$$\begin{aligned} \frac{d}{dt} \int_M F' \log(F) \frac{\omega^n}{n!} &= \int_M F'' \log(F) \frac{\omega^n}{n!} + \int_M \frac{(F')^2}{F} \frac{\omega^n}{n!} \\ &= -\frac{1}{2} \int_M (F')^2 \rho'(F) \log(F) \frac{\omega^n}{n!} \\ &\quad + \frac{1}{2} \frac{\int_M \chi^{-1}(F) \log(F) \frac{\omega^n}{n!}}{\int_M \chi^{-1}(F) \frac{\omega^n}{n!}} \int_M (F')^2 \rho'(F) \frac{\omega^n}{n!} + \int_M \frac{(F')^2}{F} \frac{\omega^n}{n!}, \end{aligned}$$

where $\rho = \log \chi$. For now, we are able to find an example of such a ρ for which the sign of the expression (4.8) is non negative. It suffices to choose

$$\rho(x) = \log \log(x+1).$$

Then for any $x > 0$, $\rho' = \frac{1}{(x+1) \log(x+1)} > 0$ and $\chi = \log(x+1) > 0$. So

$$(4.9) \quad \frac{1}{x} - \frac{1}{2} \rho'(x) \log x = \frac{1}{x} - \frac{1}{2} \frac{\log x}{(x+1) \log(x+1)} > 0.$$

Therefore, $\mathcal{E}(F)$ is convex along the geodesic with respect to the weighted metric with weight $\chi = e^\rho$.

5. THE EXAMPLE OF THE CONSTANT WEIGHT METRIC

In this section we consider back the features listed in Section 4 for the special case given by the metric which has weight constantly one; namely, the metric gotten by plugging $\chi = 1$ into (4.1), that is

$$(5.1) \quad \langle G_1, G_2 \rangle_{F;1} := \int_M G_1 G_2 \frac{\omega^n}{n!}.$$

Remark 5.1. *At the level of Kähler metrics, the scalar product is given by*

$$\int_M (\Delta_\phi \psi)^2 \frac{\omega_\phi^n}{\omega^n} \frac{\omega_\phi^n}{n!}.$$

5.1. The Levi-Civita connection for the constant weight metric. The existence and uniqueness of the Levi-Civita connection for the family of weighted metrics (4.1) has already been proved and discussed. So, as an application of Proposition 4.3, we have

Corollary 5.2. *The formula for the Levi-Civita connection for the constant weight metric is given by*

$$(5.2) \quad D_t^1 G = \frac{d}{dt} G.$$

5.2. The Sectional Curvature for the constant weight metric. Concerning the sectional curvature, we can read off Theorem 4.7 that it is constantly zero. Moreover, Lemma 4.4 tells us that also the Riemann curvature tensor is constantly zero. Let us state this observation precisely.

Corollary 5.3. *The constant weight metric (5.1) has sectional curvature zero.*

5.3. Geodesics for the constant weight metric. We presently don't know whether most of the weighted metrics do admit geodesics. We know for sure that it is the case for the Calabi metric, and now we want to show that a geometry as rich and explicit as that one of the Calabi metric exists also for the constant weight metric. We begin with applying Proposition 4.11 to the constant weight metric.

Corollary 5.4. *The geodesic equation for the flat scalar product (5.1) is $F_{tt} = 0$. In terms of Kähler potentials, the geodesic equation is*

$$(5.3) \quad \frac{d}{dt} (\Delta_\phi \phi') + (\Delta_\phi \phi')^2 = 0.$$

The geodesic equation for the constant weight metric is thus solvable, both when we consider the Cauchy and the Dirichlet problems. Moreover, we do have smooth explicit geodesics. We summarize those properties in the next result.

Theorem 5.5. *For any given $F_0, F_1 \in \mathbb{C}$, there is a unique geodesic arc for the constant weight metric (5.1) connecting F_0 and F_1 . Such geodesic arc is given by*

$$(5.4) \quad F(t) = tF_1 + (1-t)F_0 \in \mathbb{C}, \quad t \in [0, 1].$$

For any $F \in \mathbb{C}$ and any $G \in T_F \mathbb{C}$, the geodesic line starting at F with speed G is given by

$$(5.5) \quad F(t) = F + tG,$$

where t ranges in a suitable open neighborhood of 0 such that $F(t) > 0$.

We say G is the non-negative direction if $G \geq 0$ and negative if $G < 0$. Note that the geodesic line may be degenerate along the negative directions.

5.4. Geometric properties of the constant weight metric. In this subsection, we always assume that when $C_1(M) \leq 0$. A surprising fact that we will discuss later is a convexity property of the K -energy along the geodesics of the space $(\mathcal{C}, \langle \cdot, \cdot \rangle_{F;1})$ in the cases when $C_1(M) \leq 0$. As a result, we find back the uniqueness of Kähler-Einstein metrics, in any Kähler class when the first Chern class vanishes, and in $-2\pi C_1(M)$ when the first Chern class is negative. We have to emphasize here that the uniqueness of Kähler-Einstein metrics when $C_1(M) \leq 0$ was proved by Calabi by integration by part [5].

Proposition 5.6. *Along the geodesics of the constant weight metric, the second variation of the Mabuchi energy is given by the following formula*

$$\frac{d^2\nu}{dt^2} = \int_M (\Delta_\phi \phi')^2 \frac{\omega_\phi^n}{n!} + \int_M ([\partial\bar{\partial}P - \rho(\omega_0)] * \partial\phi', \bar{\partial}\phi')_{g_\phi} \frac{\omega_\phi^n}{n!}.$$

Here above $\Delta_\phi P := \omega_\phi * \rho(0) - \underline{S}$.

Proof. The argument goes similarly to that one of Proposition 5.6. \square

The Ricci potential h_ω is defined as

$$\rho(\omega) - \lambda\omega = \frac{\sqrt{-1}}{2} \partial\bar{\partial}h_\omega.$$

Under the normalized conditions $\int_M e^{h_\omega} \omega^n = V$.

Lemma 5.7. *When ω_ϕ belongs to the first Chern class, then we have*

$$P = -\lambda\phi + h_\omega$$

Proof. Since

$$\begin{aligned} S - \lambda n &= g_\phi^{i\bar{j}} (\rho_{i\bar{j}}(\omega_\phi) - \lambda g_{\phi i\bar{j}}) \\ &= g_\phi^{i\bar{j}} (\rho_{i\bar{j}}(\omega_\phi) - \rho_{i\bar{j}}(\omega) + \rho_{i\bar{j}}(\omega) - \lambda g_{i\bar{j}} + \lambda g_{i\bar{j}} - \lambda g_{\phi i\bar{j}}) \\ &= \Delta_\phi (-\log \frac{\omega_\phi^n}{\omega^n} - \lambda\phi + h_\omega). \end{aligned}$$

Therefore we have

$$(5.6) \quad P = -\lambda\phi + h_\omega. \quad \square$$

After applying Lemma 5.7 into Proposition 5.6, we obtain

$$\frac{d^2\nu}{dt^2} = \int_M (\Delta_\phi \phi')^2 \frac{\omega_\phi^n}{n!} - \lambda \int_M |\nabla \phi'|_{g_\phi}^2 \frac{\omega_\phi^n}{n!}.$$

Suppose that $C_1(M) \leq 0$, then $\frac{d^2\nu}{dt^2} \geq 0$. We obtain that

Proposition 5.8. *When $C_1(M) \leq 0$, the K -energy is convex along geodesics of the constant weight metric.*

A direct corollary of the convexity is

$$\frac{d}{dt} \nu(\omega_\varphi)|_{t=1} \geq \frac{d}{dt} \nu(\omega_\varphi)|_{t=0}.$$

We assume that ω_0 and $\omega_1 = \omega_\phi$ are two Kähler-Einstein metrics in the same Kähler class. Then, write $\omega_1 = \omega_0 + i\partial\bar{\partial}\psi$. Since $\int_M \phi' \omega_\phi^n = 0$, the above equation

implies that $\phi' = 0$, and thus the geodesic is trivial and hence $\omega_0 = \omega_1$. Hence, we arrive at the alternative proof of Calabi's uniqueness theorem. The energy and the distance from ω to ω_φ w.r.t. the constant weight metric are

$$E(\omega, \omega_\varphi) = \int_M (\Delta_\varphi \varphi')^2 \left(\frac{\omega_\varphi^n}{\omega^n}\right)^2 \omega^n,$$

$$d(\omega, \omega_\varphi) = \int_0^1 \sqrt{E(\omega, \omega_\varphi)} dt.$$

Theorem 5.9. *When $C_1(M) \leq 0$, the Kähler-Einstein metric is unique.*

Definition 5.1. We introduce a new functional which is the square norm of the scalar potential f under the normalization condition $\int_M e^f \frac{\omega_\varphi^n}{n!} = 1$,

$$G(\omega_\varphi) = \int_M f^2 \frac{\omega_\varphi^n}{n!}.$$

In particular, in the canonical class, f is the Ricci potential $f = -\log \frac{\omega_\varphi^n}{\omega^n} + h_\omega - \lambda\varphi$. We list the properties of G in the canonical class.

- (1) The Euler-Lagrange equation is $(\Delta_\varphi + \lambda)(f \frac{\omega_\varphi^n}{\omega^n})$. When $\lambda \leq 0$, $f = 0$. So the critical points are the Kähler-Einstein metrics.
- (2) At a Kähler-Einstein metric, the second variation

$$G'' = \frac{1}{2} \int_M [(\Delta_\varphi + \lambda)\varphi']^2 \omega^n \geq 0.$$

The equality holds if and only if φ' vanishes.

- (3) The gradient flow is of the form

$$f' = -f.$$

Now we emphasis on the gradient flow of the G -functional: $f' = -f$ with smooth initial Kähler potential $\varphi(0) = \varphi_0$. Solving this ODE, we have $f = e^{-t}f_0$ for $f_0 = -\log \frac{\omega_{\varphi_0}^n}{\omega^n} + h_\omega - \lambda\varphi_0$. Now the new equation is a Monge-Ampère equation with the right hand side depends on t . I.e.

$$\frac{\omega_\varphi^n}{\omega^n} = e^{h_\omega - \lambda\varphi - e^{-t}f(0)} \doteq e^F.$$

The advantage of this flow is that the term e^{-t} is always bounded. The apriori estimats achieve from the following steps. When $C_1 < 0$, C^0 -estimate follows from the maximum principle. At the maximum point p , we have $\varphi = -h_\omega + \frac{\omega_\varphi^n}{\omega^n} + e^{-t}f(0) \leq -h_\omega + e^{-t}f(0)$ which is bounded from above when t is large. The lower bound follows in the same way. When $C_1 = 0$, C^0 -estimate follows from Yau's estimate [29].

The second order estimate also follows from Yau's paper [29]. Choosing C such that $C + \inf_{i \neq k} R_{i\bar{i}k\bar{k}} = 1$, at the point p where $e^{-C\varphi}(n + \Delta\varphi)$ achieve the maximal value, we have

$$\begin{aligned} 0 &\geq \Delta'(e^{-C\varphi}(n + \Delta\varphi))(p) \\ &\geq e^{-C\varphi} \{-Cn(n + \Delta\varphi) + (C + \inf_{i \neq k} R_{i\bar{i}k\bar{k}})(n + \Delta\varphi)^{\frac{n-F}{n-1}} e^{\frac{-F}{n-1}} + \Delta F - n^2(\inf_{i \neq k} R_{i\bar{i}k\bar{k}})\}(p). \end{aligned}$$

Here $\Delta F = \Delta(h_\omega - \lambda\varphi - e^{-t}f(0))$. So the second order estimate follows by combining the C^0 -estimate.

The $C^{2,\alpha}$ -estimate follows from Evans-Krylov estimate, for the concrete constant dependence is specified in Proposition 4.6 in Calamai-Zheng [9]. So we revisit the existence theorem of Kähler-Einstein metric when $C_1(M) \leq 0$ which was proved by the continuity method of Yau [29] and by the Kähler-Ricci flow by Cao [10].

Theorem 5.10. *When $C_1(M) \leq 0$, the gradient flow of the G -functional has long time existence and converges to a Kähler-Einstein metric.*

In the following we discuss the general properties of the distance function, K -energy and the G -functional in any Kähler class.

Proposition 5.11. *The following geometric inequalities hold.*

(1) *The energy inequality:*

$$(5.7) \quad \nu(\omega_\varphi) - \nu(\omega) \leq d(\omega_\varphi, \omega) \sqrt{G(\omega_\varphi)} .$$

(2) *The geodesic distance has lower bound:*

$$d(\varphi_0, \varphi_1) \geq \max \left\{ -V^{-\frac{1}{2}} \int_{\Delta_{\varphi_0} \varphi'_0 < 0} \Delta_{\varphi_0} \varphi'_0 \omega_{\varphi_0}^n, V^{-\frac{1}{2}} \int_{\Delta_{\varphi_1} \varphi'_1 > 0} \Delta_{\varphi_1} \varphi'_1 \omega_{\varphi_1}^n \right\}$$

(3) *Triangle inequality: $d(\varphi_0, \varphi_1) - d(\varphi_1, \varphi_2) \leq d(\varphi_0, \varphi_2) \leq d(\varphi_0, \varphi_1) + d(\varphi_1, \varphi_2)$.*

Proof. Proof of (1). We first connect ω_φ and ω by the smooth geodesic. Since

$$\frac{d}{dt} \nu(\omega_\varphi) = - \int_M \Delta_\varphi \varphi' \frac{\omega_\varphi^n}{\omega^n} f \omega^n ,$$

then the proposition follows from the Cauchy-Schwarz inequality

$$\begin{aligned} \nu(\omega_\varphi) - \nu(\omega) &= - \int_0^1 \int_M \Delta_\varphi \varphi' \frac{\omega_\varphi^n}{\omega^n} f \omega^n dt \\ &\leq d(\omega_\varphi, \omega) \sqrt{G(\omega_\varphi)} . \end{aligned}$$

Proof of (2). From the geodesic equation $\frac{d}{dt} (\Delta_\varphi \varphi') = - (\Delta_\varphi \varphi')^2 \leq 0$, we have

$$\Delta_{\varphi_1} \varphi'_1 \leq \Delta_{\varphi(t)} \varphi'(t) \leq \Delta_{\varphi_0} \varphi'_0 .$$

Then the energy E at $t = 1$ is bounded by below by the Schwarz inequality,

$$\sqrt{E(1)} \geq V^{-\frac{1}{2}} \int_{\Delta_{\varphi_1} \varphi'_1 > 0} \Delta_{\varphi_1} \varphi'_1 \omega_{\varphi_1}^n .$$

Similarly, we have at $t = 0$,

$$\sqrt{E(0)} \geq -V^{-\frac{1}{2}} \int_{\Delta_{\varphi_0} \varphi'_0 < 0} \Delta_{\varphi_0} \varphi'_0 \omega_{\varphi_0}^n .$$

Since $\partial_t E = 0$ along the geodesic, we have $\sqrt{E(t)} \geq \max\{\sqrt{E(0)}, \sqrt{E(1)}\}$. Thus the proposition follows since $d = \int_0^1 \sqrt{E(t)} dt$.

Proof of (3). The triangle inequality follows from the flat curvature property directly. \square

Choosing the path $\varphi(t) = t\varphi$, we have

Corollary 5.12. *If there is a positive constant C such that $\frac{1}{C} \leq \Delta\varphi \leq C$ and $G(\omega_\varphi) \leq C$, then the K -energy of ω_φ is bounded from above by a constant which depends on C_1 .*

In particular, when λ is zero, the upper bound of $\nu(\omega_\varphi)$ only depends on $\frac{1}{C} \leq \Delta\varphi \leq C$. When λ is nonzero, it depends on both $\frac{1}{C} \leq \Delta\varphi \leq C$ and $\int_M \varphi^2 \omega^n$.

Corollary 5.13. *The K -energy has lower bound when M admits a constant scalar curvature Kähler metric.*

Proof. In (5.7) choose ω_φ to be a constant scalar curvature Kähler metric and ω to be any Kähler metric. Then this properties follows from $G(\omega_\varphi) = \nu(\omega_\varphi) = 0$. \square

5.5. The metric space structure; two non-isometric flat metrics. We are going to see that the space $(\mathcal{C}, \langle \cdot, \cdot \rangle_{F;1})$ plays as a counterpart of the space $(\mathcal{H}, \int_M \psi^2 \frac{\omega^n}{n!})$. This is to say, both the spaces are flat, but a pleasant feature is that they are not isometric, so that we can interpret $(\mathcal{C}, \langle \cdot, \cdot \rangle_{F;1})$ (via the isometric model given by the Monge-Ampère map) as another flat structure on the space of Kähler metrics \mathcal{H} .

Now that we observed that the scalar product (5.1) is flat, it is natural to address the following question. Consider, on the space of Kähler metrics \mathcal{H} , the trivial scalar product given by

$$(5.8) \quad \langle \psi_1, \psi_2 \rangle_\phi := \int_M \psi_1 \psi_2 \frac{\omega^n}{n!}.$$

The independence of the right hand side on subscript ϕ appearing on the left hand side turns the above scalar product into a flat one. Moreover, we have just seen that this other scalar product on \mathcal{H} is flat as well

$$(\psi_1, \psi_2)_\phi = \int_M (\Delta_\phi \psi_1) \cdot (\Delta_\phi \psi_2) \frac{\omega_\phi^n}{\omega^n} \frac{\omega_\phi^n}{n!}.$$

The question is whether the two scalar products are isometric or not. To answer the question, we want to compare the distance functions induced by the two flat structures. In the case of the scalar product (5.8), for any two (normalized) Kähler potentials ϕ_0 and ϕ_1 , there is a unique geodesic connecting them, which is given by

$$\phi(t) = (1-t)\phi_0 + t\phi_1,$$

with $t \in [0, 1]$. The fact that this geodesic is contained in \mathcal{H} follows from the Euclidean-convexity of that space. The distance function induced by that metric is the same as the length of a geodesic arc. Namely, the distance between ϕ_0 and ϕ_1 for this structure, which we label as $d_{\text{trivial}}(\phi_0, \phi_1)$ is

$$(5.9) \quad d_{\text{trivial}}(\phi_0, \phi_1) = \int_M (\phi_1 - \phi_0)^2 \frac{\omega^n}{n!}.$$

In order to get the distance function for the scalar product (5.1), we look at the expression (5.4) in Theorem 5.5; then, we recall that under the Monge-Ampère diffeomorphism, there exist two Kähler metrics ϕ_0 and ϕ_1 such that $F_k = \frac{\omega_{\phi_k}^n}{\omega^n}$ with $k = 0, 1$. Thus, the expression (5.4) of the geodesic arc can be rewritten, in the space \mathcal{H} , as

$$\frac{\omega_{\phi(t)}^n}{\omega^n} = (1-t) \frac{\omega_{\phi_0}^n}{\omega^n} + t \frac{\omega_{\phi_1}^n}{\omega^n}.$$

A corollary of Theorem 5.5, together with the argument of [8, Lemma 3], which proves that geodesics are length minimizing, give the following fact

Proposition 5.14. *The constant weight metric endows the space \mathcal{C} with a metric structure. The length of a geodesic arc gives the distance between its endpoints.*

We label the distance in this second case as $\text{dist}(\phi_0, \phi_1)$, and we get the expression

$$\text{dist}(\phi_0, \phi_1) = \int_M \left(\frac{\omega_{\phi_1}^n - \omega_{\phi_0}^n}{\omega^n} \right)^2 \frac{\omega^n}{n!}.$$

To summarize the above discussion, we state the following result

Proposition 5.15. *The space of Kähler metrics \mathcal{H} endowed with the Riemannian structure (5.1) is not isometric to the same space endowed with the Riemannian structure (5.8).*

We are going to discuss the metric completion of the constant weight metric. It is apparent that the constant weight metric is a L^2 metric with respect to the measure induced by the fixed Kähler metric ω . Thus, it is immediate to get the following formula

Proposition 5.16. *The metric completion space for the constant weight metric is given by*

$$\overline{\mathcal{C}}^{L^2(M, \omega)} = \{F \in L^2(M, \omega) \mid F \geq 0 \text{ a.e.}\}.$$

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